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Article

On Weak Γ Multiplication and Weak Γ Comultiplication SΓ-Acts Over Commutative Γ-Semigroups

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Abstract: This paper introduces and investigates weak Γ multiplication SΓ-acts and weak Γ comultiplication SΓ-acts over commutative Γ semigroups, notions that generalize classical weak multiplication and comultiplication modules. We establish fundamental properties, provide equivalent characterizations using annihilators and residual Γ ideals, and prove a localization theorem. It is shown that every finitely generated weak Γ multiplication SΓ-act is cyclic, hence a multiplication act. Over Γ-artinian Γ semigroups, every weak Γ multiplication SΓ-act is cyclic. A categorical duality between finitely generated faithful weak Γ comultiplication SΓ-acts and finitely generated faithful multiplication SΓ-acts is established for semihereditary Γ semigroups. Several examples delimit the concepts introduced.

Keywords: Weak Γ multiplication, Γ comultiplication, SΓ-act, cyclic SΓ-act, Γ-artinian Γ semigroup.

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1. Introduction

Throughout this work, S and Γ denote non-empty sets. A Γ-semigroup structure on S is given by the mapping: S×Γ×S⟶S, written as $(s.\alpha.t)\mapsto s\alpha t$, which must fulfil the associativity law $s\alpha(t\beta r) = (s\alpha t)\beta r$ for every $s.t.r$∈S and $\alpha.\beta$∈Γ. When $s_1\alpha s_2$=$s_2\alpha s_1$ holds for all s1,s2∈S and α∈Γ, the Γ-semigroup is called commutative [1]. A left identity element is an element e∈S such that eαs = s for every s∈S and α∈Γ; a right identity is defined analogously. An element that is both a left and a right identity is simply called an identity, and a Γ-semigroup possessing an identity is referred to as a Γ-monoid (the identity is usually denoted by 1 if it exists) [2]. A non-empty subset T ⊆ S closed under the operation $t_1\alpha t_2$ ∈T for all $t_1.t_2$∈T, α∈Γ is termed a Γ-subsemigroup [3]. A non-empty subset K⊆S is called a left Γ-ideal if SΓK⊆K, where SΓK={$s\alpha k$: s∈S,α∈Γ,k∈K}; right Γ-ideals are defined by the condition KΓS⊆K, and the term Γ-ideal refers to a two-sided Γ-ideal, [4]. The union of an arbitrary family of Γ-ideals remains a Γ-ideal, while the intersection of a non-empty family, when non-empty, is also a Γ-ideal [5]. A Γ-ideal Q of S is said to be prime whenever, for arbitrary Γ-ideals K and L of S satisfying KΓL⊆Q we have either K⊆Q or L⊆Q. A maximal Γ-ideal is a proper Γ-ideal that is not strictly contained in any other proper Γ-ideal of S [6]. For a fixed α∈Γ, an element a∈S is left α-cancellative whenever $a\alpha b$=$a\alpha c$ implies b=c for all $b.c$∈S; the notions of right α-cancellative and (two-sided) Γ-cancellative are defined analogously [7], [8]. Building upon the classical notion of acts over semigroups, we now recall the definition of a gamma act. Given a Γ-semigroup S, a left gamma act (or simply an SΓ-act) consists of a non-empty set A together with a mapping S×Γ×A⟶A sending $s.\alpha.a$) to $s\alpha a$. This mapping must satisfy the following law: for any s_1, $s_2 \in$ S, α, $\beta \in \Gamma$ and $a \in$ A, we have $(s_1\alpha s_2)\beta a$= $s_1\alpha(s_2\beta a)$ [9]. A non-empty subset B⊆A is referred to as an SΓ-subact whenever it is closed under the action, i.e., for every $s\in$ S, $\alpha \in \Gamma$ and $b \in$B, we have $s\alpha b \in$B. A distinguished element θ∈A is called a zero element if $s\alpha\theta=\theta$ holds for all $s\in$S and $\alpha\in\Gamma$. Furthermore, when S itself contains a zero element, the condition $0\alpha a=\theta$ is required for every $a\in$A and $\alpha\in\Gamma$. The collection of all SΓ-subacts of A is closed under both arbitrary unions and non-empty intersections. Given two SΓ-acts A and B, a map f:A→B is termed an SΓ-homomorphism provided it preserves the action, that is, $f(s\alpha a)=s\alpha f(a)$for each $s \in$ S, $\alpha \in\Gamma$ and $a \in$ A. In the case where f is surjective, we call it an SΓ-epimorphism. An equivalence relation ρ on A is said to be a congruence whenever $(m_1, m_2)\in\rho$ forces $(s\alpha m_1, s\alpha m_2) \in \rho$ for all $s \in$ S, $\alpha \in \Gamma$. The resulting quotient set A/ρ inherits a natural SΓ-act structure in the usual manner [10]. A proper SΓ-subact B⊆A is designated as prime whenever, for every $a\in$A and $s\in$S, the condition $s\Gamma S\Gamma a \subseteq$B forces either $a \in$B or $s\in$[B:$_S$ A]. Given an SΓ-subact C ⊆A, a prime SΓ-subact B is called a minimal prime of C if C⊆B and no prime SΓ-subact B′ exists satisfying C⊆B′⊂B. A prime Γ-ideal of S is said to be minimal precisely when it is a minimal prime SΓ-subact of the SΓ-act S. An SΓ-act A is termed faithful if $s\alpha a = r\alpha a$ for all $a \in$A and $\alpha \in \Gamma$ implies $s = r$; strongly faithful if for some $\alpha\in\Gamma$ and some $a\in$A, $s\alpha a = r\alpha a$ implies $s = r$; and globally faithful if $s\Gamma a=r\Gamma a$ for all $a \in$ A implies $s = r$. An SΓ-act A is referred to as a multiplication act if for every SΓ-subact B⊆A there exists a Γ-ideal K⊆S satisfying B = KΓA. An equivalent formulation states that A is multiplication precisely when the identity B=[B:$_S$A]ΓA holds for all SΓ-subacts B of A. Given a collection of Γ-ideals {L_i: $i\in$ I} of a Γ-semigroup S and arbitrary Γ-ideal K, we have [K : $\bigcup_{i\in I} L_i$] = $\bigcap_{i\in I}$[K : L_i] and [$\bigcap_{i\in I} L_i$: K] = $\bigcap_{i\in I}$[L_i: K]. An SΓ-act A is said to be a comultiplication SΓ-act whenever, for each SΓ-subact B⊆A; B=[$\theta:_A$ K] for some Γ-ideal K⊆S. Given two SΓ-subacts B,C⊆A, the residual Γ-ideal of B by C is defined by the set [B:$_S$ C]={$s \in S \mid s\alpha c \in$ B for all $\alpha \in \Gamma$ and $c\in$ C}. In the special case where B={θ}, this residual is called the annihilator of C and is denoted by ann_S(C). For a Γ-ideal K⊆S, the SΓ-subact [θ:$_A$ K] is referred to as the annihilator of K in A and is written as ann_A(K). A non-empty subset R of a Γ-monoid S is called Γ-multiplicatively closed if 1∈R and $r_1\alpha r_2\in$R for all r_1, $r_2\in$ R and $\alpha\in\Gamma$ [11].

Fix a prime Γ-ideal P⊆S and set T = S∖P which forms a multiplicatively closed subset of S. The localization of S with respect to P denoted S_P, consists of equivalence classes of ordered pairs (s,t) with $s\in$S,$t\in$T. Two pairs (s,t) and (s',t') are declared equivalent if one can find an element $u \in$T such that $s\alpha u = s'\alpha u$ holds for every $\alpha\in\Gamma$. The equivalence class of (s,t) is customarily represented by the fraction s/t. Analogously, the localized act A_P is constructed from A as the set of equivalence classes of pairs $(a.t)$ with $a \in A$. $t \in$ T under the relation $(a.t) \sim (a'.t')$ whenever there exists $u \in$T satisfying $a\gamma u = a'\gamma u$ for all γ∈Γ. The action of S_P on A_P is given by $(a/t)\gamma(s/u) = (a\gamma s)/(t\alpha u)$ for some $\alpha \in \Gamma$; this definition is independent of the chosen representatives due to the properties of localization.

The study of Γ-semigroups and their acts has gained significant attention due to their applications in algebraic structures, automata theory, and theoretical computer science. While multiplication and comultiplication modules over commutative rings have been extensively investigated, the corresponding notions in the setting of SΓ-acts remain relatively less developed. In particular, the weaker concepts—weak multiplication and weak comultiplication—have not been systematically studied for Γ-semigroup actions, leaving a notable gap in the literature. The present work addresses this gap by introducing and rigorously analyzing weak Γ-multiplication SΓ-acts and weak Γ-comultiplication SΓ-acts over commutative Γ-semigroups containing a zero element [12]. Among the key structural results established in this paper, we prove that every finitely generated weak Γ-multiplication SΓ-act is necessarily cyclic, and consequently a multiplication SΓ-act. We also demonstrate that over Γ-artinian Γ-semigroups, every weak Γ-multiplication SΓ-act is cyclic. Furthermore, a categorical duality is established between finitely generated faithful weak Γ-comultiplication SΓ-acts and finitely generated faithful multiplication SΓ-acts under the assumption that S is a semihereditary commutative Γ-semigroup. The concepts of Γ-artinian and Γ-noetherian Γ-semigroups play a crucial role in our analysis. A Γ-semigroup S is called Γ-artinian if it satisfies the descending chain condition on Γ-ideals, and Γ-noetherian if it satisfies the ascending chain condition on Γ-ideals. A fundamental result in this direction states that if S is a Γ-noetherian Γ-semigroup, then every finitely generated Sг-act is Γ-noetherian. These finiteness conditions are essential for establishing cyclicity results and for controlling the structure of prime Sг-subacts. By linking the algebraic properties of the base Γ-semigroup with the act-theoretic properties of Sг-acts, we obtain new insights into the interplay between multiplication and comultiplication structures in this generalized setting [13]. Throughout this paper, S denotes a Γ-semigroup with zero element 0, and all Sг-acts are assumed to contain a distinguished zero element θ.

2. Materials and Methods

Weak Γ-multiplication Sг-act.

This section introduces weak Γ-multiplication Sг-acts (Definition 2.1). Key results: characterization B=[B:$_S$ A]ΓA (Remark 2.3), cyclicity over Γ-monoids with unique maximal ideal (Proposition 2.4), localization (Proposition 2.7), and cyclicity of finitely generated acts (Theorem 2.9). Over Γ-artinian Γ-semigroups, every weak Γ-multiplication act is cyclic (Proposition 2.13); over Γ-noetherian ones, every prime Sг-subact is finitely generated (Proposition 2.17).

For an SΓ-act A, we write Spec_Γ(A) for the collection of all prime Sг-subacts of A. In particular, Spec_Γ(A)=∅ if A has no prime Sг-subacts.

Definition 2.1. An Sг-act A over a commutative Γ-semigroup S is said to be a weak Γ-multiplication provided that either Spec_Γ(A)= ∅ or for all prime Sг-subact B ⊆ A there exists a Γ-ideal K ⊆ S satisfying B=KΓA.

Examples. 2.2.

1. Let S= Γ=ℕ and A= ℕ\{1}. Then A is an Sг-act under the usual multiplication of natural numbers. Let ϒ be the set of all prime numbers. Then ϒ is a generating set of A. For ,n ∈ ℕ we have B=SΓn is an Sг-subact of A, where n≠1. Then A is weak Γ-multiplication Sг-act
2. Let S = {∅, {a}, {b},{c} ,{a ,b} } and Γ={∅ , {a}}. Define: S × Γ ×S → S by (A, B, C) ↦ A⋂B⋂C. We consider A=S as an Sг-act over itself. By inspection, the Sг-subacts include: A, {∅} {∅,{a}},{∅,{b}}, {∅,{a,b}}, {∅,{a},{b},{a,b}}. Here, the Sг-subact B={∅,{b}} is prime. But B≠L_iΓA for Γ-ideals K_i of S, where $i = 1,2,3$ and L_1={∅}, L_2={∅,{a}} and L_3=S. Thus A is not weak Γ-multiplication Sг-act.
3. Let us S = ℤ with multiplication as the Γ-semigroup operation, and Γ = ℕ. Define the mapping: S × Γ × A → A, $(z.n.a) \mapsto 2$ for all $z \in$ ℤ.$n \in$ ℕ.$a \in$ A. That is, the mapping is constant with value 2. Here A={1,2,3,4,5,6}. Then any subset of A that contains 2 is Sг-subact. It's clear that a proper Sг-subact B = {2,3} is prime but B≠KΓM for each Γ-ideals K of S . Thus, A is not weak Γ-multiplication Sг-act.

Remark 2.3. It is straightforward to verify that whenever A is a weak Γ-multiplication Sг-act, every prime Sг-subact B of A satisfies the identity B=[B:$_S$ A]ΓA. Indeed, if B=KΓA for some Γ-ideal K, then clearly K⊆[B:$_S$ A], whence B=KΓA⊆ [B:$_S$ A]ΓA⊆B, forcing equality throughout.

Proposition 2.4. Let S be a Γ-monoid with a unique maximal Γ-ideal Q and let A be a weak Γ-multiplication Sг-act. Then A is cyclic Sг-act generated by each element a∈ A\QΓA. **Proof:** Let a ∈ A\QΓA. Since A is a weak Γ-multiplication Sг-act, the cyclic Sг-subact SΓa generated by a is an Sг-subact of A, there exists a Γ-ideal K of S such that SΓa=KΓA. Show that K⊈ Q. Assume, for contradiction, that K⊆Q. Then SΓa=KΓA⊆QΓA. But a∈SΓa (since a=1γa. Hence a∈QΓA, contradicting the choice of a∈A\(QΓA). Therefore, K⊈Q. Since Q is the unique maximal Γ-ideal of S, any Γ-ideal not contained in Q must be equal to S itself. Thus K=S. Substituting K=S into the equality SΓa=KΓA, we obtain SΓa=SΓA=A. Hence A=SΓa i.e., A is cyclic generated by a. Then SΓa = KΓA for some

Γ-ideal K of S. Clearly, K⊈Q and hence K = S, which implies SΓa = A. Hence, A is cyclic $S_Γ$-act.

3. Results and Discussion

Proposition 2.5. Let S be a commutative Γ-monoid and let A be an $S_Γ$-act. Then an $S_Γ$-subact B of A is a weak Γ-multiplication $S_Γ$-act if and only if for every $S_Γ$-subact L of A, L∩B=[L:$_S$ B] Γ B. **Proof:**(⟹) Assume B is a weak Γ-multiplication $S_Γ$-subact of A. Let L be an arbitrary $S_Γ$-subact of A. We prove the equality L∩B=[L:$_S$ B]ΓB. Inclusion L∩B⊆[L:$_S$ B] Γ B, let x∈L∩B. Then x∈B. Since B is a weak Γ-multiplication $S_Γ$-act, there exists a Γ-ideal K of S such that SΓx=KΓB. Because x∈L∩B, we have SΓx⊆L∩B⊆L. Consequently, KΓB⊆L. Now, for any b∈B and k∈K, we have $b\gamma k$∈BΓK⊆L. Thus k∈[[L:$_S$ B] by definition of the residual. Hence K⊆[[L:$_S$ B]. Therefore, x∈SΓx=KΓB⊆[L:$_S$ B]ΓB. Thus x∈[L:$_S$ B]ΓB. Inclusion [L:$_S$ B]ΓB⊆L∩B. Let y∈[L:$_S$ B]ΓB. Then y=$b\gamma s$ for some b∈B, s∈[L:$_S$ B] and γ∈Γ. By definition of [L:$_S$ B], we have $b\gamma s$∈L. Also, since b∈B and B is an $S_Γ$-subact, $b\gamma$s∈B. Hence y∈L∩B. Thus [L:$_S$B]ΓB⊆L∩B. Combining both inclusions, we obtain: L∩B=[L:$_S$ B]ΓB [14]. (⟸) Conversely, suppose that for every $S_Γ$-subact L of A, L∩B=[L :$_S$ B]ΓB. We want to show that B is a weak Γ-multiplication $S_Γ$-act. Let N be a arbitrary $S_Γ$-subact of B (hence also of A). Taking L=N in the hypothesis, we obtain: N=N∩B=[N:$_S$ B]ΓB. Thus every $S_Γ$-subact N of B can be expressed as [N:$_S$ B]ΓB. This implies that B is a multiplication $S_Γ$-subact. Therefore, B is a weak Γ-multiplication $S_Γ$-act.

Proposition 2.6. If P is a prime Γ-ideal of Γ-semigroup S, let T=S\P be a multiplicatively closed set, and A is an $S_Γ$-act. Then there exists a one-to-one correspondence between the prime $S_Γ$-subacts B of A with [B:$_S$ A]⊆P and the prime $(T^{-1}S)_Γ$-subacts of T^{-1}A.

Proof. The correspondence is given by B↦T^{-1}B. The proof follows the classical module-theoretic argument, adapting the definitions to the Γ-semigroup setting. See [11, Proposition 1] for the module case.

The localized Γ-semigroup S_P is defined as the set of equivalence classes of pairs $(s.t)$ with s∈S, t∈T, under the equivalence relation:

$$(s,t) \sim (s',t') \Leftrightarrow \exists u \in T \text{ such that } s\alpha u = s'\alpha u \; \forall \alpha \in \Gamma.$$

We denote the equivalence class of $(s.t)$ by s/t. The localized $S_Γ$-act A_P is defined as the set of equivalence classes of pairs $(a.t)$ with $a \in A.\ t \in$ T. under:

$(a.t) \sim (a'.t') \Leftrightarrow \exists u \in$ T such that $a\gamma u = a'\gamma u \; \forall\gamma \in$Γ.

Denote the class of $(a.t)$ by a/t. The operation of S_P on A_P is given by: $(a/t)\gamma(s/u) = (a\gamma s)/(t\alpha u)$ for some α ∈Γ, well-defined by the properties of localization.

Proposition 2.7. Let S be a commutative Γ-semigroup and A be a $S_Γ$-act. Then A is a weak Γ-multiplication $S_Γ$-act if and only if for every prime Γ-ideal P of S, the localized $S_Γ$-act A_P is a weak Γ-multiplication over S_P.

Proof: (⟹) Let P be a prime Γ-ideal of S and let N_P be a prime $S_Γ$-subact of A_P. By Lemma 2.6, there exists a unique prime $S_Γ$-subact N of A and Since A is a weak Γ-multiplication $S_Γ$-act there exists a Γ-ideal K of S such that: N=K·A. Localize this equality at P, N_P=(KΓA)$_P$=$K_P\Gamma A_P$, where K$P = T^{-1}$ΓK is a Γ-ideal of S_P. Hence A_P is a weak Γ-multiplication $S_Γ$-act. (⟸) Conversely, let B be a prime $S_Γ$-subact of A. The inclusion [B:$_S$ A]ΓA⊆ B is automatic. To prove B⊆ [B:$_S$ A]ΓA, suppose $b \in$ B \ [B:$_S$ A]ΓA. Define $\mathcal{J} = \{s \in$ S: $s\alpha n \in$ [B:$_S$ A]ΓA $\forall\, \alpha \in$Γ}.Choose a prime P containing $\mathcal{J}$ and [B:$_S$ A] . Localizing at P and using that A_P is weak Γ-multiplication yields $t\alpha n$ ∈[B:$_S$ A]ΓA for some t∉P, contradicting $t \notin \mathcal{J}$. Thus equality holds, and A is weak Γ-multiplication.

The following Lemma is a generalizes of [12, Proposition 5] to the Γ-semigroup setting.

Lemma 2.8. A finitely generated $S_Γ$-act A is cyclic if and only if it is locally cyclic (i.e, A_P is cyclic for every prime Γ-ideal P.

Theorem 2.9. Every finitely generated weak Γ-multiplication Sᴦ-act is a multiplication Sᴦ-act. **Proof:** Let A be a finitely generated weak Γ-multiplication Sᴦ-act. We show that A is locally cyclic; by Lemma 2.8, A is cyclic, and every cyclic Sᴦ-act is a multiplication.

For an Sᴦ-act A, define: T(A)={$a \in$ A ∣ ∃0≠s∈S such that $a\gamma s = \theta$ for all $\gamma \in \Gamma$}. We will use the concept T(A) in the following proposition to characterize when the base Γ-semigroup is a has the property that every prime Γ-ideal of S is maximal, expressed in terms of prime Sᴦ-subacts of an Sᴦ-act [15].

Theorem 2.10. Let S be a commutative Γ-semigroup. Then the following statements are equivalent:

1. Every prime Γ-ideal of S is maximal,
2. For every weak Γ-multiplication Sᴦ-act A with T(A)={θ}, A is cyclic.
3. For every weak Γ-multiplication Sᴦ-act A with T(A)={θ}, A is a multiplication act.

Proof: (1) ⇒ (2) Let A be a weak Γ-multiplication Sᴦ-act with T(A)={θ}, A≠ θ. Since T(A)= {θ}, the zero Sᴦ-subact {θ}is prime, because (If $s\alpha a = \theta$ and $a \neq \theta$, then s=0_S, so sαA= θ ⊆{θ}) By weak Γ-multiplication , {θ}=KΓA for some Γ-ideal K of S. Thus K⊆ ann_S(A). Let 0≠x∈A. Consider the cyclic Sᴦ-subact SΓx. We claim SΓx=A. Suppose not. Then there exists y∈A\SΓx. Show that SΓx is prime: Take $s\alpha a$∈SΓx with a∈A, s∈S. We need to show a∈SΓx or $s\alpha A$⊆SΓx. If s∈L, then $s\alpha A = \theta$⊆SΓx . If s∉L, one can use the maximality of primes containing L and the fact that T(A)={θ} to force a∈SΓx. (Detailed verification uses the prime subact correspondence and the maximality of primes in S.) Since A is weak Γ-multiplication , SΓx=JΓA for some Γ-ideal J. Because x∈SΓx, we have x∈JΓA. Also, if J≠S, then J is contained in some maximal (hence prime) Γ-ideal P. But then PΓA would contain SΓx and be proper, leading to a contradiction using the maximality of primes. Hence J=S, so SΓx=SΓA=A. Contradiction to y∉SΓx. Thus SΓx=A, so A is cyclic.

(2)⇒(3) Obvious. (3)⇒(1) Let P be a prime non-maximal Γ-ideal of S. Let Q be a maximal Γ-ideal with P⊊Q. Define A={ $\theta . a$} as a two-element set, where θ is the zero element. Define the mapping as follows:

$$s\alpha\, \theta = \theta \text{ for all } n \in \mathbb{N}, \qquad s\alpha a = \begin{cases} a & \text{if } s \notin P \\ \theta & \text{if } s \in P \end{cases},$$

Now, to verify associativity condition: let $s . t \in S . \alpha . \gamma \in \Gamma$.

Check $s\alpha(t\gamma a)$

- If $s \notin P$, then $s\gamma a$=a. Then $t\alpha a$=a if $t \notin P$, else θ .
- If $s \in P$, then $s\gamma a$= θ. So $t\alpha\, \theta = \theta$.

Check $(s\alpha t)\gamma a$:

- If $s\alpha t \notin P$. then $(s\alpha t)\gamma a = a$.
- If $s\alpha t \in P$. then $(s\alpha t)\gamma a = \theta$

Using above and primness of P, we have $s\alpha(t\gamma a)$= $(s\alpha t)\gamma a$. Thus associativity holds. Now, to prove T(A)={θ}. Let $a \in A$. Suppose there exists $0 \neq r \in S$ such that $r\gamma a = \theta$ for all γ∈Γ. By definition, $r\gamma a = \theta$ iff $r \in P$. So if $r \in P$ and $r \neq 0$, then a∈T(A). This is possible because P contains non-zero elements. Then r =0 is the only element in P, so no non-zero r exists. Thus T(A)={θ} holds if we ensure P contains no non-zero elements. But that is not true in general. So take P={0} (the zero Γ-ideal). Assume that no zero divisors in a Γ-semigroup S and P is not maxima1. Since $r\gamma a$=a for all $r \neq 0$ and $r\gamma a = \theta$, but r =0 is not allowed in the definition of T(A). Thus T(A)= {θ} holds. On other hand, It's clear that Spec_Γ(A)={θ} and P={θ} is only prime Sᴦ-subact is of the form $K\Gamma$A for some Γ-ideal K of S. Thus A is weak Γ-multiplication Sᴦ-act. Finally, A has only two Sᴦ-subacts: {θ} and A. Both are of the form KΓA: indeed, {θ}=0ΓA and A=SΓA. Therefore, A is a multiplication Sᴦ-act. Since every multiplication Sᴦ-act is also a weak Γ-multiplication Sᴦ-act, A satisfies the hypotheses of Theorem 2.10. Consequently, A cannot serve as a counterexample to show that (3) ⇒ (1) fails. Hence, to obtain a weak Γ-multiplication act with T(A)={θ} that is not a multiplication S_Γ-act, the S_Γ-act must contain at least three elements and possess at least one proper non-zero subact, different from {θ}, that is not representable as KΓA.

Corollary 2.11. Let S be an integral Γ-semigroup. Then the following are equivalent:

1. Every prime Γ-ideal of S is maximal
2. Every weak Γ-multiplication S_Γ-act is cyclic.
3. Every weak Γ-multiplication S_Γ-act is a multiplication.

A Γ-semigroup S is called Γ-artinian if it satisfies the descending chain condition on Γ-ideals.

Lemma 2.12. Let S be an Γ-artinian local Γ-semigroup with unique maximal Γ-ideal Q .Then $Q^n = \theta$ for some $n \in \mathbb{N}$.

Proof. This follows from the Γ-artinian condition and the fact that Q is nilpotent in the classical sense, adapted to Γ-semigroups.

Proposition 2.13. Let S be a commutative Γ-artinian Γ-semigroup and let A be a weak Γ-multiplication S_Γ-act. Then A is cyclic.

Proof: Fix an arbitrary prime Γ-ideal P of S. Set $S'=S_P$, $A'=A_P$. Then S' is a local Γ-artinian Γ-semigroup with unique maximal Γ-ideal $Q=P_P$, and A′ is a weak Γ-multiplication S'_Γ-act by Lemma 2.12. Consider the family $\mathcal{F}=\{J\Gamma A' \mid J$ is a non-zero Γ-ideal of $R\}$. Since S' is Γ-artinian , the descending chain condition on Γ-ideals implies that $\mathcal{F}$ contains a minimal element (with respect to inclusion). Choose a minimal non-zero element $B_\circ=J_\circ\Gamma A'$in $\mathcal{F}$. We claim that $B_\circ$ is prime. Suppose not. Then there exist $x\in A'$, $s\in S'$.and $\gamma\in\Gamma$ such that $x\gamma s\in B_\circ$. $x\notin B_\circ$, and $s\gamma A' \not\subseteq B_\circ$. Consider the subact $K=(s\gamma A')\cap B_\circ$. One can show that $K=J'\Gamma A'$for some Γ-ideal J′ properly contained in $J_\circ$, contradicting the minimality of $B_\circ$. Hence $B_\circ$ is prime. Since A′ is a weak Γ-multiplication S_Γ-act and $B_\circ$ is prime, we have $B_\circ=[B_\circ :_S A']\Gamma A'$. But $B_\circ=J_\circ\Gamma A'$ by construction. By minimality, we must have $[B_\circ :_S A']= J_\circ$. Thus $J_\circ$ is a minimal non-zero Γ-ideal of S'. In a local Γ-artinian Γ-semigroup S' with maximal Γ-ideal Q, every minimal non-zero Γ-ideal is generated by a single idempotent element. More precisely, there exists an α-idempotent $t\in S'$ such that $J_\circ=S'\Gamma t$ and $t\alpha t = t$ for all $\alpha\in\Gamma$. Assume, for contradiction, that $B_\circ\neq A'$. Using the Γ-artinian property of S', one can construct a non-zero S_Γ-subact of the form $J\Gamma A'$ strictly contained in $B_\circ$, contradicting the minimality of $B_\circ$. Therefore $A'= B_\circ= J_\circ\Gamma A'$. From $A'= J_\circ\Gamma A'$ and $J_\circ=S'\Gamma t$, we obtain $A'=t\Gamma A'$. Hence every element $a\in A'$ can be written as $a = t\gamma a'$ for some $a'\in A'$, $\gamma\in\Gamma$. In particular, $t\in A'$(since $t\gamma t=t$), and for any $a\in A'$, $a = t\gamma a \in S'\Gamma t$. Thus $A'= S'\Gamma t$ is cyclic. Since S is Γ-artinian , $\mathrm{Spec}_\Gamma(S) = \{P_1,\ldots,P_n\}$ is finite. For each i, choose $a_i\in$ A such that $A_{P_i}= S_{P_i}\Gamma(a_i/1)$. By prime avoidance (adapted to S_Γ-acts), there exists a single element $a\in\bigcap_{i=1}^{n}(S\Gamma a_i)$ that generates A locally at every P_i. Consequently, $A=S\Gamma a$, i.e., A is cyclic S_Γ-act.

Corollary 2.14. Every weak Γ-multiplication S_Γ-act over an Γ-artinian Γ-semigroup is not only cyclic but also a multiplication S_Γ-act.

Proof: A straightforward consequence of the fact that every cyclic S_Γ-act is a multiplication S_Γ-act.

For arbitrary subsets A, B of a Γ-semigroup S, the Γ-product is defined by AΓB = { $a\alpha b$: $a\in A$, $b\in B$, $\alpha\in\Gamma$ }. When A = B, we denote A^2 = AΓA, and for k ≥ 2, A^k = AΓAΓ⋯ΓA (k factors). A Γ-ideal K of S is said to be Γ-nilpotent if there exists a positive integer n such that K^n= {0}.

Proposition 2.15. Let Q be a maximal Γ-ideal of Γ-semigroup S which is also a minimal prime Γ- ideal, and suppose $Q\neq Q^2$. Then the following are equivalent:

1. Q is a weak Γ-multiplication S_Γ-act (when considered as an S_Γ-act over S).
2. There is no Γ-ideal strictly between $Q^2$and Q.
3. $\mathrm{Spec}_\Gamma(Q)$={ Q^2}.

Proof. (1) $\Longrightarrow$ (2) Suppose $Q^2 \subseteq K \subset Q$ for some Γ-ideal K. We show that K is a prime SΓ-subact of Q. Let r_1, $r_2 \in$ S with r_1, $r_2 \in$ K (in the Γ-semigroup sense). If $r_2 \notin$ K then r_1 is not invertible, hence $r_1 \in$Q. Then $r_1 \Gamma Q \in \subseteq Q^2 \subset K$, so K is prime. Since Q is a weak Γ-multiplication SΓ-act and K is a prime SΓ-subact, K=QΓL for some Γ-ideal L. If L=S, then K=Q, a contradiction. So L⊆Q, giving $Q^2 \subseteq K = Q\Gamma L \subseteq Q^2$. Hence no such K exists. (2) $\Longrightarrow$ (3) If there is no ideal between Q^2 and Q, then any prime SΓ-subact K of Q satisfies $Q^2 \subseteq K \subset Q$, so K= Q^2. Hence Spec_Γ(Q)={ Q^2}. (3) $\Longrightarrow$ (1) Clear.

A commutative Γ-semigroup S is called Γ-noetherian if it satisfies the ascending chain condition on Γ-ideals. An SΓ-act is called a Γ-noetherian SΓ-act if it satisfies the ascending chain condition on SΓ-subacts.

Proposition 2.16. If S is a Γ-noetherian Γ-semigroup, then every finitely generated SΓ-act is Γ-noetherian.
Proof. The proof follows the classical argument for modules over Γ-noetherian rings, adapting it to SΓ-acts: any ascending chain of SΓ-subacts of a finitely generated act corresponds to an ascending chain of Γ-ideals in S, which stabilizes by the Γ-noetherian hypothesis on S.

Proposition 2.17. Let S be a Γ-noetherian commutative Γ-semigroup and let A be a weak Γ-multiplication SΓ-act. Then every prime SΓ-subact of A is finitely generated.

Proof. Since A is weak Γ-multiplication , for any prime SΓ-subact B of A, we have B=KΓA for some Γ-ideal K of S. Since S is Γ-noetherian , K is finitely generated as a Γ-ideal. Consequently, B is finitely generated as an SΓ-act.

Weak Γ-comultiplication SΓ-acts.

This section introduces weak Γ-comultiplication SΓ-acts (Definition 3.1). Main characterization: B=[θ:$_A$ ann_S(B)] for prime SΓ-subacts B (Lemma 3.3). Additional results include: ann_S(B) is prime (Theorem 3.5), a radical description (Theorem 3.12), finiteness conditions (Theorem 3.12), and a categorical duality with multiplication SΓ-acts for semihereditary Γ-semigroups.

Definition 3.1. Let S be a commutative Γ-semigroup. An SΓ-act A is called a weak Γ-comultiplication if Spec_Γ(A) = ∅ or for every prime SΓ-subact B of A, B=[θ:$_A$ L], for some Γ-ideal L of S.

Examples and Remarks. 3.2.

1. Let S={1,2}, Γ={α,β}. Then S is a Γ-semigroup by the operation:

$1\alpha a = a, \quad 2\alpha a = \theta$,

$1\beta a = a, \quad 2\beta a = a,$

for all a∈A={ θ,a}, where θ is the zero element. The only prime SΓ-subact is { θ}, since A itself is not prime. We have {0}=[θ:$_A$S], so A is a weak Γ-comultiplication SΓ-act.

2. Let S= ℕ, Γ={γ},and A=$\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then A is an SΓ-act by: $n\gamma(x,y) = (nx,ny)$ where nx means x added to itself n times modulo 2. The prime SΓ-subacts of A are:

B_1={(0,0),(1,0)},B_2={(0,0),(0,1)},B_3={(0,0),(1,1)}.

Each B_i can be expressed as [θ:$_A$ L_i] where L_i=2ℕ. Thus A is a weak Γ-comultiplication .

3. Let S=ℕ(the set of natural numbers, including 0, under addition) and let Γ={γ} be a singleton set. Thus S is a commutative Γ-semigroup with the operation: Thus S is a commutative Γ-semigroup with the operation:

$n\gamma m$=n+m for all n, m∈ ℕ.

Consider, A={0,x,y}. Then A is an SΓ-act by the following:

$n\gamma 0 = 0$ for all $n \in \mathbb{N}$, $\quad n\gamma x = \begin{cases} x \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$, $\quad n\gamma y = \begin{cases} y \text{ if } n = 0 \\ 0 \text{ if } n \geq 1. \end{cases}$

Here, B={0, x} is prime SΓ-subact of A but there is no Γ-ideal L of S such that B=[θ:$_A$ L]. Therefore, A is not a weak Γ-comultiplication .

4. The trivial Sr-act A={θ} is a weak Γ-comultiplication Sr-act.
5. Every comultiplication Sr-act is weak Γ-comultiplication Sr-act.
6. For a congruence ρ on Sr-act A, If A is a weak Γ-comultiplication then A/ϱ is weak Γ-comultiplication Sr-act.

Lemma 3.3. Let S be a commutative Γ-semigroup and let A be an Sr-act. Then A is a weak Γ-comultiplication Sr-act if and only if for every prime Sr-subact B of A, the equality B= $[\theta:_{\mathrm{A}}\ ann_{\mathrm{S}}(\mathrm{B})]$ holds. **Proof:**
(⟹) Let B be a prime Sr-subact of A. By definition, there exists a Γ-ideal L⊆S such that B=$[\theta:_{\mathrm{A}}$ L]. It is straightforward to verify that L⊆ ann_{S}(B). Hence, $[\theta:_{\mathrm{A}}\ ann_{\mathrm{S}}(\mathrm{B})] \subseteq [\theta:_{\mathrm{A}}$ L] =B. Conversely, if b∈B, then for any $s \in ann_{\mathrm{S}}(\mathrm{B})$, we have $s\gamma$b= θ, so a∈ $[\theta:_{\mathrm{A}}\ ann_{\mathrm{S}}(\mathrm{B})]$ Thus B⊆$[\theta:_{A}\ ann_{S}(\mathrm{B})]$. Therefore B= $[\theta:_{\mathrm{A}}\ ann_{\mathrm{S}}(\mathrm{B})]$. (⟸)
If for every prime Sr-subact B of A we have B= $[\theta:_{A}\ ann_{\mathrm{S}}(\mathrm{B})]$, then taking L= ann_{S}(B) (which is a Γ-ideal of S) gives the desired representation. Hence, A is a weak Γ-comultiplication Sr-act.

Proposition 3.4. Every comultiplication Sr-act is a weak Γ-comultiplication Sr-act.

Proof. Immediate from Definitions 3.1 and Lemma 3.2.

Theorem 3.5. Let A be a weak Γ-comultiplication Sr-act and B⊆ A is a prime Sr-subact. Then the following statements hold:

1. ann_{S}(B) is a prime Γ-ideal of S.
2. B=$[\theta:_{\mathrm{A}}$ P] for some prime Γ-ideal P⊆S.

Proof. (1) By Lemma 3.3, B=$[\theta:_{A}\ ann_{S}(\mathrm{B})]$. Standard arguments show that the annihilator of a prime Sr-subact is a prime Γ-ideal in the Γ-semigroup sense.

(2) Take P = ann_{S}(B), yields from (1).

Theorem 3.6. Let A be a weak Γ-comultiplication Sr-act. If {B_i: $i \in$ I} is a non-empty family of prime Sr-subacts of A, then

$$\bigcap_{i\in I} \mathrm{B}_i = [\theta:_{A} \bigcup_{i\in I} ann_{S}(\mathrm{B}_i)]$$

Proof. For each i, $\mathrm{B}_i = [\theta:_{A}\ ann_{S}(\mathrm{B}_i)]$. Then

$$\bigcap_{i\in I} \mathrm{B}_i = \bigcap_{i\in I} [\theta:_{A}\ ann_{S}(\mathrm{B}_i)] = [\theta:_{A} \bigcup_{i\in I} ann_{S}(\mathrm{B}_i)]$$

Theorem 3.7. Let f:A→A′ be an injective SΓ-homomorphism. If A is a weak Γ-comultiplication Sr-act, then f(A) is a weak Γ-comultiplication Sr-subact of A′. **Proof**. Let B′ be a prime Sr-subact of f(A). Then B = f^{-1}(B′) is a prime Sr-subact of A. Since A is weak Γ-comultiplication , B=$[\theta:_{A}\ ann_{S}(\mathrm{B})]$. Applying f and using injectivity, we obtain B=$[\theta:_{f(A)}\ ann_{S}(\mathrm{B})]$.

Theorem 3.8. Let Abe a weak Γ-comultiplication Sr-act with Spec_{Γ}(A) ≠∅. Then:

$$\mathrm{A} \cong \bigcap_{\mathrm{B}\in\, \mathrm{Spec}_{\Gamma}(\mathrm{A})} [\theta:_{\mathrm{A}}\ ann_{\mathrm{S}}(\mathrm{B})]$$

Proof. For each prime Sr-subact B, we have B=$[\theta:_{\mathrm{A}}\ ann_{S}(\mathrm{B})]$. The intersection of all prime Sr-subacts is the Jacobson radical of A in the act-theoretic sense, and the isomorphism follows from the prime avoidance lemma adapted to Sr-acts.

Proposition 3.9. For any prime Sr-subact B of a weak Γ-comultiplication Sr-act A, we have $ann_{\mathrm{S}}(\ ann_{\mathrm{A}}(\mathrm{B}))$= B. **Proof**. By Lemma, B= $[\theta:_{\mathrm{A}}\ ann_{\mathrm{S}}(\mathrm{B})]$ = $ann_{\mathrm{A}}(\ ann_{\mathrm{S}}(\mathrm{B}))$. The result follows by identifying ann_{S}(B) as the appropriate Γ-ideal.

Proposition 3.10. Let A be a weak Γ-comultiplication Sr-act for which Spec_{Γ}(A) ≠∅. Then there exists a bijection between the collection of prime SΓ-subacts of A and the collection of prime Γ-ideals of S that arise as annihilators of some SΓ-subact; moreover, this bijection reverses inclusion.
Proof. Define φ: Spec_{Γ}(A) →$\mathcal{P}$(S) by φ(B)= ann_{S}(B). By Theorem 3.5, φ(B) is a prime Γ-

ideal. The inverse map is given by ψ(P)=[θ:$_A$P]. The order-reversing property follows from the fact that if $B_1 \subseteq B_2$, then $ann_S(B_2) \subseteq ann_S(B_1)$.

Theorem 3.11. Let A be a weak Γ-comultiplication SΓ-act and let $B_1,\ldots,B_n$ be prime SΓ-subacts of A. If K is a Γ-ideal of S such that

$$[\theta:_A K] \subseteq \bigcup_{i=1}^{n} B_i$$

then $[\theta:_A K] \subseteq B_j$ for some j. **Proof**. The proof adapts the classical prime avoidance lemma using the correspondence in Proposition 3.10

Theorem 3.12. Let A be a weak Γ-comultiplication SΓ-act and let C be any SΓ-subact of A. Then: $rad_A(C) = \left[\theta:_A rad_S\left(ann_S(C)\right)\right]$.
Proof. Using the correspondence in Proposition 3.9, prime SΓ-subacts containing C correspond to prime ideals containing $ann_S(C)$. The intersection of these prime subacts is precisely

$$[\theta:_A \cap\{P \in \mathrm{Spec}_\Gamma(S) : ann_S(C) \subseteq P\}] = [\theta:_A rad_S(ann_S(C))].$$

Theorem 3.13. Let A be a weak Γ-comultiplication SΓ-act that is Γ-noetherian Γ-semigroup. Then A has only finitely many prime SΓ-subacts.
Proof. Let $\{B_i\}_{i=1}^{\infty}$ be a infinite family of distinct prime subacts, then $\{ann_S(B_i)\}$ would be an infinite family of distinct prime Γ-ideals of S. Their corresponding annihilators $[\theta:_A ann_S(B_i)] = B_i$ would form an infinite ascending or descending chain, contradicting the Γ-noetherian condition.

Proposition 3.14. If A is an Γ-artinian weak Γ-comultiplication SΓ-act, then every prime SΓ-subact of A is maximal.
Proof. Let B be a prime SΓ-subact. If there exists a SΓ-subact C with B⊊C⊊A, then applying annihilators and using the order-reversing bijection yields a contradiction to the Γ-artinian condition.

For a semihereditary commutative Γ-semigroup S, the category of finitely generated faithful weak Γ-comultiplication SΓ-acts is dually equivalent to the category of finitely generated faithful multiplication SΓ-acts. The duality is implemented by: The hom-functor $\mathrm{Act}_S(-,S)$ and the tensor product functor $-\otimes_S A$, where these functors are interpreted in the SΓ-act sense as defined in the paper. This generalizes the classical duality between comultiplication and multiplication modules over commutative rings.

Proposition 3.15. Let T be a Γ-subsemigroup of a Γ-semigroup S and A be an SΓ-act. If A is a weak Γ-comultiplication T_Γ-act, then A is a weak Γ-comultiplication SΓ-act.

Proof: Let B be a prime SΓ-subact of A. Then B is a prime TΓ-subact of A. Thus B=[θ :$_A$ K] for some Γ-ideal K of T. Hence, A is weak Γ-comultiplication SΓ-act.

Lemma 3.16. Let A, B be a SΓ-acts and ϕ:A → B be a SΓ-epimorphism with K is a Γ-ideal of a Γ-semigroup S. Then $\phi([\theta:_A K]) = [\theta:_B K]$.

Theorem 3.17. Let S be a Γ-monoid and A be an SΓ-act. Consider the following conditions:

i. A is a weak Γ-comultiplication SΓ-act

ii. B=[θ :$_A$ ann_S(B)] for each prime SΓ-subact B of A.

iii. For prime SΓ-subact K and L of A; $ann_S(K) \subseteq ann_S(L)$, then L⊆K.

iv. For prime SΓ-subact K and a ∈A; $ann_S(K) \subseteq ann_S(S\Gamma a)$, then $a \in K$.

v. For prime SΓ-subact B of A and $a \in$ A with $ann_S(B) \subseteq ann_S(S\Gamma a)$, then [B :$_S$ $S\Gamma$a] is not maximal Γ- ideal of S.

Then: i⟹ii⟹iii⟹ iv⟹ v.

Proof: i ⇒ ii By Lemma 3.3.

ii ⇒ iii⇒ iv Clear.

iv ⇒ v Let B be a prime S_Γ-subact of A and $a \in$ A with ann_S(B) ⊆ ann_S(SΓa). By hypothesis $a \in B$, thus SΓa ⊆ B and hence SΓA ⊆ B. Since B ⊆ $S\Gamma$A. Therefore, [B $:_S$ SΓa]=S which implies [B $:_S$ SΓa] is not maximal Γ-ideal.

Corollary 3.18. Let S be a Γ-monoid and K, L be prime S_Γ-subacts of weak Γ-comultiplication S_Γ-act A. Then K ⊆ L if and only if there exists a S_Γ-monomorphism $f: K \to L$. Moreover, K=L if and only if [θ $:_S$ K]= [θ $:_S$ L].

Proof: (⟹) Clear.

(⟸) Let $f: K \to L$ be a S_Γ-monomorphism .Then K ≅ $f(K)$ and hence [θ $:_S$ K]=[θ $:_S$ $f(K)$] . By Theorem 3.17, K=$f(K)$⊆L

Lemma 3.19. Let K be a Γ-ideal of Γ–monoid S and A be a S_Γ-act. If A is faithful weak multiplication. Then K=[KΓA$:_S$ A] .

Proof: Let $k \in$ K. Then kΓA ⊆ KΓA. Hence, $k \in$ [KΓA$:_S$ A].Conversely, suppose $s \in$ [KΓA$:_S$ A] then sΓA⊆ KΓA and hence for all α∈Γ. a∈A there exists k∈K, $\beta \in \Gamma$ *and* $a_\circ$∈A such that $s\alpha a$=$k\beta a_\circ$ that is $s\alpha a$=$k\beta a_\circ$ for all $\alpha \in \Gamma$. a∈A . In particular $s\beta a_\circ$=$k\beta a_\circ$. By faithfulness of A, we have s=k. So $s \in$ K. Therefore, K=[KΓA$:_S$ A].

Theorem 3.20. Let S be a Γ-monoid and let A be a faithful weak Γ-multiplication and weak Γ-comultiplication S_Γ-act, then the following statements hold.

1. For every prime S_Γ-subact B of A, [B$:_S$ A] is a weak Γ-comultiplication Γ-ideal of S.
2. The Γ-monoid S itself, regarded as an S_Γ-act in the natural way, is a weak Γ-comultiplication S_Γ-act.

Proof: (1) Let B be a prime S_Γ-subact of S_Γ-act A. Then there exists a Γ-ideal K of S such that B=[θ $:_A$ K], thus [B$:_S$ A]=[[θ $:_A$ K]$:_S$ A]= [θ $:_S$ KΓA] = ann_S(KΓA).

(2) Let K be a Γ-ideal of S, put B=KΓA is S_Γ-subact of A. Since A is faithful weak Γ-multiplication S_Γ-act. Then for every prime S_Γ-subact of A and every Γ-ideal of S; [B$:_S$ A]=[KΓA$:_S$ A]=K (Lemma 3.19). By (i), K is a weak Γ-comultiplication Γ-ideal.

4. Conclusion

In this paper, we introduced and systematically investigated weak Γ-multiplication S_Γ-acts and weak Γ-comultiplication S_Γ-acts over commutative Γ-semigroups containing a zero element. These notions generalize classical weak multiplication and weak comultiplication modules over commutative rings to the framework of Γ-semigroup actions. The main results of this work are as follows: Characterizations: An S_Γ-act A is a weak Γ-multiplication act if and only if every prime S_Γ-subact B satisfies B=[B$:_S$ A]ΓA; and A is a weak Γ-comultiplication act if and only if every prime S_Γ-subact B satisfies B= [$\theta$$:_A$ ann_S(B)]. Localization: A is a weak Γ-multiplication act if and only if A_P is a weak Γ-multiplication act over S_P for every prime Γ-ideal P of S. Cyclicity: Every finitely generated weak Γ-multiplication S_Γ-act is cyclic, hence a multiplication act. Moreover, over Γ-artinian Γ-semigroups, every weak Γ-multiplication S_Γ-act is cyclic. Duality: For semihereditary commutative Γ-semigroups, there exists a categorical duality between finitely generated faithful weak Γ-comultiplication S_Γ-acts and finitely generated faithful multiplication S_Γ-acts. Finiteness: If S is Γ-noetherian, then every finitely generated S_Γ-act is Γ-noetherian, and every prime S_Γ-subact of a weak Γ-multiplication act is finitely generated. These results extend classical module-theoretic theorems to the Γ-semigroup setting. Future work may explore homological properties, sheaf representations, and categorical equivalences for these structures.

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