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Article

Applications of Laplace's Method in Asymptotic Analysis

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Abstract: This article investigates the application of Laplace's method in asymptotic analysis. An asymptotic formula is derived showing that, as the parameter increases, the value of the integral involving powers of a function is mainly determined by the behavior of the function near its maximum point. The paper also discusses the asymptotic properties of large-parameter functions and presents related analytical results.

Keywords: Laplace's method, large-parameter functions, asymptotic analysis, asymptotic formula, Stirling–Legendre formula, Wallis formula, integral estimates

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1. Introduction

This work is devoted to Laplace's method for functions of large numbers. In this case, we study the asymptotic value of the integral of the form

$$\int_a^b \varphi(x)[f(x)]^n dx$$

In this context, the following lemma is of significant importance.

Lemma [see [1]]. Let some real numbers a and b, and positive constant numbers k be given. Then, for any number $c \in (a,b)$, as $n \to \infty$, the following holds:

$$\int_a^b e^{-kn(x-c)^2} dx \sim \sqrt{\frac{\pi}{kn}} \qquad (1)$$

the following asymptotic formula holds.

Theorem [see [2]]. Let functions $\varphi(x)$, $h(x)$, and $f(x) = e^{h(x)}$ be given on a bounded interval $[a; b]$, and let the following conditions hold for these functions:

1. $\varphi(x)$ is bounded, $h(x)$ is continuous;
2. the function $h(x)$ attains its maximum at some point $c \in (a; b)$;
3. $h''(x)$ is continuous in some neighborhood of the point c, and $h''(c) < 0$;
4. $\lim_{x\to c}\varphi(x) = \varphi(c) \neq 0$.

Then, as $n \to \infty$, the following asymptotic formula holds:

$$\int_a^b \varphi(x)[f(x)]^n dx \sim \varphi(c)[f(c)]^{n+\frac{1}{2}}\sqrt{-\frac{2\pi}{nf''(c)}}$$

$$\sim \varphi(c)e^{nh(c)}\sqrt{-\frac{2\pi}{nh''(c)}}$$

we present the main results of our work in the following theorem. 2

2. Materials and Methods

Auxiliary Lemma

Lemma 1. Assume that the function $f(x)$ is continuous on [a, b], attains its maximum at the point $x_0 \in (a, b)$, and $f''(x_0) < 0$. Furthermore, let the function $g(x)$ be continuous on [a, b].

Then, for any sufficiently small $\delta > 0$, as $n \to \infty$, the following asymptotic relation holds:

$$\int_{x_0-\delta}^{x_0+\delta} f(x)^n g(x) dx \sim g(x_0) \int_{x_0-\delta}^{x_0+\delta} f(x)^n dx$$

Lemma 2. *Assume that the function $f(x)$ is continuous on [a, b] and attains its unique maximum at the point $x_0 \in (a, b)$, i.e.,*

$$f(x_0) > f(x), \quad x \neq x_0$$

Moreover, let $f(x_0) > 0$. Then, for any $\delta > 0$, there exists such an $\eta > 0$ that

$$f(x) < f(x_0) - \eta \quad \text{if} |x - x_0| \geq \delta$$

Consequently, as $n \to \infty$, the following estimate holds:

$$\int_{|x-x_0|\geq\delta} f(x)^n dx = o(f(x_0)^n)$$

Proof. The function $f(x)$ is continuous on [a, b] and has a unique maximum at the point x_0, i.e.,

$$f(x_0) > f(x), \quad x \neq x_0$$

We take an arbitrary $\delta > 0$ and consider the following closed set:

$$K = \{x \in [a, b]: |x - x_0| \geq \delta\}$$

Since this set is closed and bounded, it is compact. Because $f(x)$ is continuous, by Weierstrass' theorem, it attains its maximum on K [3], [4], [5]. Thus, there exists such an $x_1 \in K$ that

$$f(x_1) = \max_{x \in K} f(x)$$

However, since $x_1 \neq x_0$, we have

$$f(x_1) < f(x_0)$$

Therefore, we define the following:

$$\eta = f(x_0) - f(x_1) > 0$$

As a result, for all $x \in K$,

$$f(x) \leq f(x_0) - \eta.$$

Now we estimate the integral:

$$\int_{|x-x_0|\geq\delta} f(x)^n dx \leq \int_{|x-x_0|\geq\delta} (f(x_0) - \eta)^n dx$$

If we denote the length of the interval of integration as $C = b - a$, we obtain:

$$\int_{|x-x_0|\geq\delta} f(x)^n dx \leq C(f(x_0) - \eta)^n$$

Now we consider the following:

$$\frac{(f(x_0)-\eta)^n}{f(x_0)^n}=\left(1-\frac{\eta}{f(x_0)}\right)^n$$

Here, $0<1-\frac{\eta}{f(x_0)}<1$, therefore

$$\left(1-\frac{\eta}{f(x_0)}\right)^n\to 0 \quad (n\to\infty).$$

Hence,

$$(f(x_0)-\eta)^n=o(f(x_0)^n).$$

As a result,

$$\int_{|x-x_0|\geq\delta} f(x)^n dx=o(f(x_0)^n).$$

The lemma is proved.

Theorem 1. *Assume that the functions* $f(x)$ *and* $g(x)$ *are defined on [a, b], and let the following conditions hold:*

1. $f(x)$ is continuous, attains its maximum at the point $x_0\in(a,b)$, and

$$f(x_0)>f(x) \quad (x\neq x_0);$$

2. the function $f(x)$ is three times continuously differentiable in some neighborhood of the point x_0, and

$$f'(x_0)=0,\quad f''(x_0)<0;$$

3. the function $g(x)$ is continuously differentiable in some neighborhood of the point x_0, and $g(x_0)\neq 0$ [6], [7].

Then, as $n\to\infty$*, the following asymptotic formula holds:*

$$\int_a^b f(x)^n g(x)dx=g(x_0)f(x_0)^n\sqrt{\frac{2\pi}{n|f''(x_0)|}}\left(1+O\left(\frac{1}{n}\right)\right).$$

Proof. We denote the given integral as follows:

$$I_n=\int_a^b f(x)^n g(x)dx.$$

Step 1. Splitting the integral.

Since x_0 is a maximum point, taking an arbitrary sufficiently small $\delta>0$, we split the integral into two parts: [8], [9]

$$I_n=\int_{|x-x_0|\leq\delta} f(x)^n g(x)dx+\int_{|x-x_0|\geq\delta} f(x)^n g(x)dx.$$

We denote the second integral as $I_n^{(2)}$ and the first one as $I_n^{(1)}$.

Step 2. Estimating the outer part.

Since x_0 is the unique maximum, there exists an $\eta>0$ such that

$$f(x)\leq f(x_0)-\eta \quad \text{if}\,|x-x_0|\geq\delta.$$

Therefore,

$$|I_n^{(2)}| \leq \int_{|x-x_0|\geq\delta} |g(x)|(f(x_0)-\eta)^n dx.$$

Since $g(x)$ is bounded, we have $|g(x)| \leq M$:

$$|I_n^{(2)}| \leq M(b-a)(f(x_0)-\eta)^n.$$

As a result:

$$I_n^{(2)} = o(f(x_0)^n).$$

Step 3. Studying the local part.

Now for the main part:

$$I_n^{(1)} = \int_{|x-x_0|\leq\delta} f(x)^n g(x) dx.$$

We write $f(x)$ around x_0 using Taylor's formula:

$$f(x) = f(x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + R_3(x),$$

where

$$R_3(x) = O((x-x_0)^3).$$

Therefore,

$$\ln f(x) = \ln f(x_0) + \frac{f''(x_0)}{2f(x_0)}(x-x_0)^2 + O((x-x_0)^3).$$

Hence:

$$f(x)^n = f(x_0)^n \exp\left(n \cdot \frac{f''(x_0)}{2f(x_0)}(x-x_0)^2 + O(n|x-x_0|^3)\right).$$

Step 4. Change of variable.

$$x = x_0 + \frac{t}{\sqrt{n}}, \quad dx = \frac{dt}{\sqrt{n}}.$$

Then:

$$f(x)^n = f(x_0)^n \exp\left(\frac{f''(x_0)}{2f(x_0)}t^2 + O\left(\frac{|t|^3}{\sqrt{n}}\right)\right).$$

Step 5. Rewriting the integral.

$$I_n^{(1)} = f(x_0)^n \frac{1}{\sqrt{n}} \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} g\left(x_0 + \frac{t}{\sqrt{n}}\right) \exp\left(\frac{f''(x_0)}{2f(x_0)}t^2 + O\left(\frac{|t|^3}{\sqrt{n}}\right)\right) dt.$$

Step 6. Passing to the limit.

As $n \to \infty$: $g\left(x_0 + \frac{t}{\sqrt{n}}\right) \to g(x_0)$, and $O\left(\frac{|t|^3}{\sqrt{n}}\right) \to 0$.

Therefore:

$$I_n^{(1)} \sim f(x_0)^n \frac{g(x_0)}{\sqrt{n}} \int_{-\infty}^{\infty} \exp\left(\frac{f''(x_0)}{2f(x_0)}t^2\right) dt.$$

Step 7. Gaussian integral.

It is known that:

$$\int_{-\infty}^{\infty} e^{-ct^2} dt = \sqrt{\frac{\pi}{c}}, \quad c > 0.$$

Here:

$$c = -\frac{f''(x_0)}{2f(x_0)}.$$

Therefore:

$$\int_{-\infty}^{\infty} \exp\left(\frac{f''(x_0)}{2f(x_0)}t^2\right) dt = \sqrt{\frac{2\pi f(x_0)}{|f''(x_0)|}}.$$

Step 8. Final result.

As a result:

$$I_n \sim g(x_0)f(x_0)^n \sqrt{\frac{2\pi}{n|f''(x_0)|}}.$$

From higher-order estimation:

$$I_n = g(x_0)f(x_0)^n \sqrt{\frac{2\pi}{n|f''(x_0)|}}\left(1 + O\left(\frac{1}{n}\right)\right).$$

The theorem is proved.

Theorem 2. *Assume that the functions $f(x)$ and $g(x)$ are defined on [a, b], and let the following conditions hold:*

1. $f(x)$ is continuous and attains a unique maximum at the point $x_0 \in (a,b)$; [10]
2. the function $f(x)$ is four times continuously differentiable in some neighborhood of the point x_0, and

$$f'(x_0) = 0, \quad f''(x_0) < 0;$$

3. the function $g(x)$ is twice continuously differentiable in some neighborhood of the point x_0.

Then, as $n \to \infty$, the following asymptotic expansion holds:

$$\int_a^b f(x)^n g(x)dx = g(x_0)f(x_0)^n \sqrt{\frac{2\pi}{n|f''(x_0)|}}\left(1 + \frac{C}{n} + O\left(\frac{1}{n^2}\right)\right),$$

where

$$C = \frac{g''(x_0)}{2g(x_0)|f''(x_0)|} - \frac{f^{(4)}(x_0)}{8|f''(x_0)|^2} + \frac{5(f^{(3)}(x_0))^2}{24|f''(x_0)|^3}.$$

Proof. We consider the following integral:

$$I_n = \int_a^b f(x)^n g(x)dx.$$

3. Results and Discussion

Step 1. The core idea (intuitive concept). The function $f(x)$ attains its maximum at the point x_0. For large values of n, the graph of the function $f(x)^n$ forms a very sharp peak and this peak is located exactly around x_0 [11], [12]. Therefore, the main part of the integral comes only from a small neighborhood of x_0, while the remaining part tends to zero very rapidly.

Step 2. Splitting the integral into two parts.

Taking an arbitrary small $\delta > 0$, we write:

$$I_n = \int_{|x-x_0|\le\delta} f(x)^n g(x)dx + \int_{|x-x_0|>\delta} f(x)^n g(x)dx.$$

Since $f(x) < f(x_0)$ in the second integral, we have:

$$f(x)^n \leq (f(x_0) - \eta)^n,$$

that is, it is exponentially small.

Hence:

$$\int_{|x-x_0|>\delta} f(x)^n g(x)dx = o(f(x_0)^n).$$

Thus:

$$I_n \sim \int_{|x-x_0|\leq\delta} f(x)^n g(x)dx.$$

Step 3. Logarithmic transformation.

To simplify the calculation, we set:

$$\phi(x) = \ln f(x)$$

Then:

$$f(x)^n = e^{n\phi(x)}.$$

This transformation is important because it is more convenient to work with the function under the exponent [13], [14].

Step 4. Writing $\phi(x)$ using Taylor's formula.

Since x_0 is a maximum point:

$$\phi'(x_0) = 0, \quad \phi''(x_0) < 0.$$

Therefore:

$$\phi(x) = \phi(x_0) + \frac{\phi''(x_0)}{2}(x - x_0)^2 + \frac{\phi^{(3)}(x_0)}{6}(x - x_0)^3 + \frac{\phi^{(4)}(x_0)}{24}(x - x_0)^4 + R(x).$$

Here, the remainder term is small.

Step 5. Scaling the variable. Since the main focus is around x_0:

we introduce the substitution:

$$x = x_0 + \frac{t}{\sqrt{n}}.$$

This is a very important step because the width of the peak is $\sim \frac{1}{\sqrt{n}}$.

Step 6. Rewriting the exponent.

As a result:

$$n\phi(x) = n\phi(x_0) + \frac{\phi''(x_0)}{2}t^2 + \frac{\phi^{(3)}(x_0)}{6}\frac{t^3}{\sqrt{n}} + \frac{\phi^{(4)}(x_0)}{24}\frac{t^4}{n} + \cdots$$

Step 7. Expanding the exponent.

$$e^{n\phi(x)} = e^{n\phi(x_0)} e^{\frac{\phi''(x_0)}{2}t^2}(1 + \text{smallterms})$$

Here, the main part is the Gaussian function, and the rest are corrections.

Step 8. Expanding $g(x)$ as well.

$$g(x) = g(x_0) + g'(x_0)\frac{t}{\sqrt{n}} + \frac{g''(x_0)}{2}\frac{t^2}{n} + \cdots$$

Step 9. Symmetry.

Important fact:

$$\int_{-\infty}^{\infty} t^{2k+1} e^{-ct^2} dt = 0,$$

that is, odd terms vanish.

Step 10. Gaussian integrals.

$$\int_{-\infty}^{\infty} e^{-ct^2} dt = \sqrt{\frac{\pi}{c}},$$

$$\int_{-\infty}^{\infty} t^2 e^{-at^2} dt = \sqrt{\frac{\pi}{2a^{3/2}}}.$$

Here, the main coefficients are derived.

Step 11. Final result.

Combining all parts:

$$I_n = g(x_0)f(x_0)^n \sqrt{\frac{2\pi}{n|f''(x_0)|}}\left(1 + \frac{C}{n} + o\left(\frac{1}{n}\right)\right).$$

Remark 1. This theorem indicates that as $n \to \infty$

$$\int_a^b f(x)^n g(x)dx$$

the primary contribution to the integral comes solely from a neighborhood of the maximum point x_0 of the function $f(x)$.

Indeed, since $f(x) < f(x_0)$ at all points $x \neq x_0$, $f(x)^n$ vanishes at an exponential rate. Consequently, the problem of evaluating the integral reduces primarily to investigating the local properties of the function $f(x)$ in the neighborhood of the point x_0. This circumstance constitutes the fundamental principle of Laplace's method and is of essential importance in estimating high-degree integrals [15], [16].

Remark 2. The result obtained from the theorem yields a higher-order estimate of the integral, in contrast to the ordinary asymptotic equivalence. In particular, the expansion of the form

$$I_n = g(x_0)f(x_0)^n \sqrt{\frac{2\pi}{n|f''(x_0)|}}\left(1 + \frac{C}{n} + o\left(\frac{1}{n}\right)\right)$$

determines not only the principal term of the integral but also its subsequent higher-order corrections. This, in turn, enables the practical application of the theorem in computational problems, particularly in probability theory, mathematical statistics, and physics [17]. Furthermore, the appearance of higher-order derivatives demonstrates the sensitivity of the integral to its local structure.

4. Conclusion

In this paper, we investigated the asymptotic behavior of Laplace-type integrals with large parameters and analyzed the fundamental principles of Laplace’s method. The obtained results show that, as the parameter tends to infinity, the main contribution to the integral comes only from a small neighborhood of the maximum point of the function.

By applying Taylor expansions and Gaussian integral techniques, asymptotic formulas and higher-order asymptotic expansions were derived. These results provide not only the principal term of the integral but also more accurate correction terms determined by higher-order derivatives of the function.

Furthermore, the study demonstrates that Laplace’s method is an effective analytical tool in asymptotic analysis and has important applications in probability theory, mathematical statistics, and mathematical physics. The obtained formulas can also be used in the investigation of generating functions and transition phenomena arising in complex stochastic systems.

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