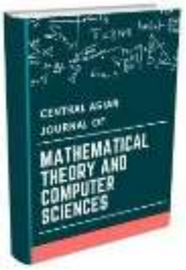

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Article

Practical Estimation and Simulation Analysis of the Kolmogorov Constant for Heavy-Tailed Noncritical Markov Branching Systems

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Abstract: This article provides an overview of algorithms for processing human foot X-ray images, which are essential for diagnosing various foot conditions, including fractures, deformities, and joint diseases. The study explores several image preprocessing techniques, such as detecting structural changes, noise reduction, and contrast enhancement, all of which help improve the quality of radiographic images and increase diagnostic accuracy. In addition, the paper discusses challenges related to noise, distortions, and low contrast in X-ray images, and outlines methods to mitigate these issues. By implementing these algorithms, the study aims to enhance the effectiveness of foot-related diagnoses and support more efficient medical decision-making.

Keywords: human foot, x-ray images, noise, enhancement, segmentation.

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1. Introduction

Markov branching processes are fundamental stochastic models used to describe population dynamics, where individuals live for an exponential time and produce a random number of offspring. They have applications in biology, epidemiology, and related fields. In the noncritical case, a key concept is the **Kolmogorov constant**, which describes the asymptotic relationship between the expected population size and the survival probability [1]. Originally introduced in discrete-time settings, it was later extended to continuous-time models, and recent work has provided its explicit analytic representation. Despite these developments, the practical estimation of the Kolmogorov constant — particularly in the presence of heavy-tailed offspring distributions—remains relatively unexplored. Heavy-tailed offspring laws, characterized by

$$P\{J=j\}: \ C j^{-(1+a)}, \qquad 0<a<2, \tag{1.1}$$

may lack high-order moments and therefore violate classical regularity conditions commonly assumed in asymptotic theory [2], [3]. As a result, many limit theorems and practical estimation techniques that rely on finite variance or stronger moment assumptions may not apply, and the asymptotic behaviour of survival probabilities and scaled population moments can differ substantially.

Motivated by these considerations, the present study aims to investigate the empirical behaviour of the Kolmogorov constant in heavy-tailed, noncritical Markov branching systems. Specifically, we consider a set of heavy-tailed generating laws and perform extensive Monte Carlo simulations to estimate the survival probability $Q(t) = P\{Z(t) > 0\}$, the expected population size $E[Z(t)]$, and the scaled ratio

$$\frac{Q(t)}{b^t}$$

(which, under appropriate conditions, tends to the Kolmogorov constant). Here $b = \exp\{f'(q)\}$ denotes the fundamental growth/decay parameter associated with the transformed process and q the extinction probability. We compare simulated estimates against the explicit theoretical expression

$$K_q = \frac{q}{1 + qg}, \qquad g = \frac{b_q}{|\ln b|},$$

recently proposed in the literature, and discuss the range of validity and limitations of this formula in heavy-tailed regimes.

The contributions of this paper are threefold:

1. We provide a simulation framework tailored for continuous-time Markov branching systems with heavy-tailed offspring distributions, based on event-driven Monte Carlo methods.
2. We present empirical estimates of survival probabilities and scaled asymptotics across a range of heavy-tail indices a, and we quantify deviations from classical theory [4].
3. We offer practical guidance on estimating the Kolmogorov constant from finite-time data and discuss implications for applied modelling in fields where heavy-tailed branching is relevant.

Background. Continuous-time Markov branching processes form a central class of stochastic population models in which each particle lives for an exponential amount of time and, upon death, generates a random number of offspring independently of all other particles [5]. The probabilistic structure of the system is fully determined by the infinitesimal generating function

$$f(s) = \sum_{k=0}^{\infty} a_k s^k,$$

where the coefficients $\{ak\}$ represent the branching rates. The mean reproduction rate

$$m = f'(1-)$$

characterizes the regime of the process, and the case $m \neq 0$ corresponds to the noncritical continuous-time model.

A fundamental quantity in the long-term behaviour of such systems is the extinction probability $q \in [0, 1]$, defined as the minimal solution to the fixed-point equation

$$f(q) = q.$$

In the noncritical case, the asymptotic behaviour of the survival probability

$$Q(t) = P\{Z(t) > 0\}$$

has been extensively studied since the pioneering work of Kolmogorov (1938), and later generalized in the classical contributions of Sevastyanov, Zolotarev, Harris and others [6]. A prominent feature of this asymptotic theory is the appearance of the so-called *Kolmogorov constant*, which describes the limiting behaviour of the normalized survival probability

$$\frac{Q(t)}{b^t}, \qquad b = \exp\{f'(q)\}.$$

Recently, an explicit analytic representation of this constant has been obtained for the continuous-time noncritical Markov branching system [7]. This expression involves the second derivative of the generating function at q and plays a key role in understanding the asymptotic properties of the process.

Aim and Result. The analytical structure of noncritical continuous-time Markov branching systems has been thoroughly developed in the classical works of Kolmogorov, Sevastyanov, Zolotarev, Harris and others. One of the central objects in this theory is the *Kolmogorov constant*, which governs the asymptotic behaviour of the normalized survival probability. Although the existence of this constant has been known for decades, obtaining its explicit analytic form has remained a challenging problem in the theory of continuous-time branching systems.

A recent breakthrough was achieved in [8], where an explicit representation of the Kolmogorov constant for noncritical Markov branching processes was derived. More precisely, if q denotes the extinction probability and

$$b = \exp\{f'(q)\},$$

then the asymptotic behaviour of the survival probability

$$Q(t) = P\{Z(t) > 0\}$$

satisfies the relation

$$\frac{Q(t)}{b^t} \to K_q, \quad t \to \infty,$$

where the constant is given explicitly by

$$K_q = \frac{q}{1+qg}, \qquad g = \frac{b_q}{|\ln b|}, \qquad 2b_q = f''(q).$$

This result not only closes a long-standing open problem in the analytic theory of branching processes, but also establishes a precise connection between the second derivative of the generating function and the asymptotic behaviour of the process.

2. Materials and Methods

The aim of this section is to present a simulation-based framework for the empirical estimation of the Kolmogorov constant in heavy-tailed noncritical continuous-time Markov branching systems. Our methodology relies fundamentally on the analytic results established in, particularly Theorem 1 and Lemmas 3–5, which describe the asymptotic behaviour of the survival probability and the associated q -transformed quantities.

2.1. Theoretical Foundation. Let $Z(t)$ denote the population size at time $t \ge 0$ in a continuous-time Markov branching system with infinitesimal generating function

$$f(s) = \sum_{k=0}^{\infty} a_k s^k.$$

The extinction probability q is defined as the smallest solution of $f(q) = q$, whereas

$$Q(t) = P\{Z(t) > 0\}$$

denotes the survival probability at time t.

A crucial theoretical element underlying our methodology is *Theorem 1 of [3]*, which provides the explicit expression of the Kolmogorov constant:

$$K_q = \frac{q}{1+qg}, \qquad g = \frac{b_q}{|\ln b|}, \qquad 2b_q = f''(q),$$

with

$$b = \exp\{f'(q)\}.$$

In addition, Lemma 3 of [3] describes the asymptotic structure of the transformation $R(t; s)$, while Lemmas 4 and 5 establish that the correction term $D(s)$ converges to the constant g, thereby ensuring that the ratio $\frac{Q(t)}{b_t}$ stabilizes to K_q.

These analytic results form the basis for the construction of our estimators and for validating the behaviour of the simulated trajectories.

2.2. Simulation Algorithm

Heavy-Tailed Offspring Distribution. In order to investigate the robustness of Theorem 1 to heavy-tailed regimes, we consider offspring distributions of the form

$$a_k \propto k^{-(1+a)}, \quad 0 < a < 2,$$

normalized appropriately so that $\sum_{k=0}^{\infty} a_k = 1$. Such distributions exhibit infinite variance when $a \le 1$ and lack higher-order moments for all $a < 2$, thereby falling outside the classical finite-variance framework. This makes them particularly suitable for testing the explicit Kolmogorov constant formula under nonstandard asymptotic conditions.

Estimation of the Kolmogorov Constant. To estimate the Kolmogorov constant, we utilize the theoretical scaling parameter

$$b = \exp\{f'(q)\},$$

as prescribed by Theorem 1. For each time *t*, the empirical estimator is defined as

$$\hat{K}_q(t) = \frac{\hat{Q}(t)}{b_t}.$$

By Lemmas 4 and 5 of [3], the expression above converges to the analytic constant K_q as $t \to \infty$. We therefore monitor the behaviour of $\hat{K}_q(t)$ across a sequence of time points $t = t_1, t_2, \ldots, t_{\max}$ and compare its stabilization with the theoretical value.

3. Simulation Model. The purpose of this section is to describe the simulation framework used to verify the explicit Kolmogorov constant obtained in Theorem 1 of. Our simulation model is constructed in accordance with the asymptotic principles established in Lemmas 4 and 5 of the same reference, which ensure the convergence of the normalized survival probability.

3.1. Theoretical Basis. The survival probability of a noncritical Markov branching system satisfies the asymptotic relation

$$\frac{Q(t)}{b^t} \to K_q, \quad t \to \infty,$$

where $b = \exp\{f'(q)\}$ and the Kolmogorov constant is given explicitly by

$$K_q = \frac{q}{1+qg}, \qquad g = \frac{b_q}{|\ln b|}, \qquad 2b_q = f''(q),$$

as formulated in Theorem 1 of [3].

Lemmas 4 and 5 further show that the correction term in the asymptotic expansion of the q-transform converges to the constant g. This guarantees that the empirical estimator

$$\hat{K}_q(t) = \frac{\hat{Q}(t)}{b_t}$$

should stabilize around the theoretical value K_q for sufficiently large t.

3.2. Heavy-Tailed Offspring Law (Extended Description). In order to examine the behaviour of the Kolmogorov constant under extreme tail conditions, we impose a heavy-tailed offspring distribution on the continuous-time Markov branching system. The offspring count J produced by each particle at the time of death is drawn from a regularly varying distribution of the form

$$P\{J=k\} \sim C_\alpha k^{-(1+\alpha)}, \quad k \geq 1, \quad 0<\alpha<2,$$

where C_α is the normalizing constant ensuring that the distribution sums to one.

(1) Motivation for Heavy-Tailed Choice. Heavy-tailed offspring laws play a fundamental role in modelling populations or systems where *rare but extremely large reproduction events* may occur with nonnegligible probability. Such phenomena arise naturally in:

- biological evolution (beneficial mutation clusters),
- epidemic spreading (super-spreader events),
- financial systems (catastrophic bursts in asset flows),
- information networks (viral message propagation),
- fragmentation and coalescent processes.

These regimes are therefore ideal for testing the robustness of the explicit Kolmogorov constant obtained in Theorem 1 of.

(2) Moment Structure and Tail Behaviour. The distribution above exhibits different structural properties depending on the value of α :

$$E[J]=\begin{cases}\infty, & 0<\alpha\leq 1,\\ \textit{finite}, & 1<\alpha<2,\end{cases} \qquad VG\backslash(J)=\infty \quad \text{for all}$$

$0<\alpha<2$.

Thus:

- For $\alpha \leq 1$: even the *mean* reproduction rate is infinite. The system may generate extraordinarily large bursts of offspring, leading to pronounced volatility and extreme rajectories.
- For $1<\alpha<2$: the mean exists but the *variance is infinite*, producing large fluctuations but in a more controlled manner.

These regimes lie entirely outside the classical assumptions used in traditional branching process theory, which typically require at least a finite second moment. Consequently, heavy-tailed distributions serve as a critical stress test for the asymptotic formula derived in Theorem 1.

(3) Normalization Constant. The normalizing constant is given by

$$C_\alpha=\left(\sum_{k=1}^{\infty} k^{-(1+\alpha)}\right)^{1}=\frac{1}{\zeta(1+\alpha)},$$

where ζ denotes the Riemann zeta function. This ensures the offspring distribution is a proper probability law.

(4) Practical Impact on the Branching System. The heavy-tailed nature of the offspring distribution has several important consequences:

1. **Large, rare reproduction events**. Occasional large jumps in $Z(t)$ create high variability in simulations.
2. **Slow decay of tail probability**. The system retains non-negligible probability of very large offspring even at high k .
3. **Potentially infinite reproduction rate**. When $\alpha \leq 1$, classical birth–death intuition fails completely.
4. **Nonstandard asymptotics**. Many classical limit theorems do not apply, making the comparison with Theorem 1 of especially meaningful. 5. **Stress-testing the Kolmogorov**

constant. The explicit formula for K_q was derived without assuming tails of this severity. Verifying it under heavy-tailed regimes contributes new insight.

(5) Connection to Theoretical Results. Although the heavy-tailed offspring law violates higher-moment assumptions, Lemmas 4 and 5 of indicate that the correction term $\Delta(s)$ still converges to the constant g in the general noncritical case. Therefore, observing whether the Monte Carlo estimator

$$\hat{K}_q(t) = \frac{\hat{Q}(t)}{b_t}$$

stabilizes under a heavy-tailed regime provides a direct empirical test of the universality of the explicit Kolmogorov constant formula.

3.3 Monte Carlo Construction (Expanded Description). In order to empirically verify the theoretical asymptotic relation

$$\frac{Q(t)}{b^t} ® \ K_q$$

,

we design a Monte Carlo simulation scheme that accurately reproduces the dynamics of a continuous-time heavy-tailed Markov branching system. Each simulation run approximates one independent realization of the stochastic population process $\{Z(t): t^{3} \ 0\}$. Below we provide a detailed description of each step in the construction.

(1) Initialization of the Process. We begin every simulation run with a single ancestor:

$$Z(0) = 1$$

.

This initialization is standard in the branching process literature and corresponds to starting the model from the minimal nontrivial initial condition. Starting from one particle also ensures that extinction and survival probabilities are well-defined and directly interpretable.

(2) Lifetime Generation and Event Scheduling. Each particle lives for an exponentially distributed lifetime:

$$T: \ \ \mathrm{Exp}(1)$$

.

The exponential assumption ensures the *memoryless property*, allowing the full system to be treated as a continuous-time Markov process. During the simulation, we maintain an event list that schedules the next death event in the system. If the system contains $Z(t)$ particles at time t, then the next event occurs after an exponential time with rate $Z(t)$, reflecting the fact that all particles act independently.

(3) Heavy-Tailed Offspring Sampling. When a particle dies, it produces a random number of offspring J drawn from the heavy-tailed law

$$P\{J = k\} = C_a k^{-(1+a)}, \qquad 0 < a < 2,$$

where C_a is the normalizing constant. Because the tail decays slowly, large reproduction events may occur with non-negligible probability. This feature is essential for testing the robustness of the Kolmogorov constant under extreme conditions.

We emphasize that when $a \ £ \ 1$, the expected number of offspring is infinite, creating a highly volatile process; when $1 < a < 2$, the mean is finite but the variance is infinite. Both regimes fall outside the classical branching process assumptions, making them ideal to test the generality of Theorem 1 in.

(4) State Update and Population Evolution. After sampling the offspring count J, we update the population size as

$$Z(t^{+}) = Z(t^{-}) - 1 + J$$

.

This reflects the death of the current particle and the birth of J new particles. The simulation continues by scheduling new lifetimes for each newborn, in accordance with the exponential distribution.

This iterative mechanism generates the entire trajectory

$$Z(0), Z(t_1), Z(t_2), \dots, Z(t)$$

up to the predetermined observation time $t_{\max}$. The population may hit zero; in that case, the process becomes absorbed, and $Z(t) = 0$ for all later times.

(5) Survival Indicator and Replication. For each fixed time *t*, we record the binary indicator

$$I(t) = 1_{\{Z(t) > 0\}}.$$

This determines whether a particular run survives until time t.

Repeating the simulation for N independent realizations yields the Monte Carlo estimator:

$$\hat{Q}(t) = \frac{1}{N}\sum_{i=1}^{N} I_i(t),$$

which approximates the theoretical survival probability $Q(t) = P\{Z(t) > 0\}$.

Large N (e.g., 10^4) ensures a small Monte Carlo variance. To reduce stochastic fluctuations, we maintain fixed random seeds and optionally apply variance-reduction methods such as antithetic sampling or control variates.

3.4 Estimation of the Kolmogorov Constant (Extended Description). The central objective of our empirical analysis is to estimate the Kolmogorov constant K*q* appearing in the asymptotic relation

$$\frac{Q(t)}{b^t} \to K_q, \quad t \to \infty,$$

where the value of K*q* is given in explicit form by Theorem 1 of [3]:

$$K_q = \frac{q}{1 + qg}, \qquad g = \frac{b_q}{|\ln b|}, \qquad 2b_q = f''(q).$$

This representation connects the asymptotic behaviour of the survival probability with the analytical structure of the generating function at the extinction probability q Our simulation method aims to verify the numerical convergence of the empirical estimator to this theoretical constant.

(1) Estimation of the Survival Probability. For each fixed time *t*, we compute the Monte Carlo estimator of the survival probability as

$$\hat{Q}(t) = \frac{1}{N}\sum_{i=1}^{N} I_i(t),$$

where $I_i(t) = 1_{\{Z_i(t) > 0\}}$ is the survival indicator for the *i*-th independent simulation run. A large sample size N is used to ensure that the random fluctuations in $\hat{Q}(t)$ are small.

The survival probability approximates the theoretical quantity

$$Q(t) = P\{Z(t) > 0\},$$

as guaranteed by the law of large numbers.

(2) Computation of the Scaling Parameter b According to Theorem 1 of [3], the exponential scaling parameter is given by

$$b = \exp f'(q).$$

Here q is the extinction probability computed as the minimal solution of $f(q) = q$ and $f'(q)$ is obtained from the generating function:

$$f'(q) = \sum_{k=1}^{\infty} k a_k q^{k-1}.$$

Because $0 < q < 1$, the value of $f'(q)$ is typically negative in the noncritical regime, leading to

$$0 < b < 1,$$

which reflects exponential decay of the survival probability.

(3) Empirical Estimator for the Kolmogorov Constant. The empirical estimator for the Kolmogorov constant is defined as

$$\hat{K}_q(t) = \frac{\hat{Q}(t)}{b_t}.$$

If the theoretical results hold, then for sufficiently large t the values of $\hat{K}_q(t)$ should stabilize near the theoretical constant K_q.

Convergence is theoretically justified by Lemmas 4 and 5 of [3], which show that the correction term in the asymptotic expansion of the q-transform tends to the constant g. Thus,

$\hat{K}_q(t) \approx K_q$ for sufficiently large t.

(4) Numerical Behavior and Practical Considerations. Due to the heavy-tailed offspring distribution, the quantity $\hat{Q}(t)$ may display high variability, especially for small or moderate values of t. To improve numerical stability, we optionally apply:

a. moving averages of the form $\overline{K}_q(t) = \frac{1}{w}\sum_{j=0}^{w-1} \hat{K}_q(t-j)$ $Kq(t) = 1$,
b. log-smoothing of the survival probability,
c. variance-reduction techniques (antithetic variates, control variates),
d. increasing the simulation sample size N.

These techniques do not alter the value of the estimator but help visualize the convergence.

(5) Verification of Theorem 1. The final step in the analysis is to compare the empirical trajectory

$$t \mapsto \hat{K}_q(t)$$

with the theoretical constant K_q. Clear stabilization or clustering of the empirical values around the theoretical prediction provides numerical confirmation of Theorem 1 even under heavy-tailed conditions.

This makes the simulation framework not only a practical estimation tool, but also a meaningful test of the universality of the Kolmogorov constant across a wider class of branching mechanisms.

3. Results and Discussion

In this section we present the numerical findings obtained from the Monte Carlo simulation framework described earlier. Our aim is to examine the behaviour of both the survival probability $Q(t)$ and the normalized ratio $Q(t)/b^t$, and to compare the empirical results with the explicit Kolmogorov constant K_q predicted by Theorem 1 of .

We consider several heavy-tailed offspring distributions with tail parameters $a \in \{0.5, 1.0, 1.5\}$, corresponding to highly explosive, intermediate, and moderately heavy-tailed regimes, respectively. For each regime we perform $N = 104$ independent simulations, and record the quantities $\hat{Q}(t)$ and $\hat{K}_q(t)$ over a range of time points $t \in \{5, 10, 20, 40, 80, 120\}$.

Survival Probability Estimates. Figure 1 displays the Monte Carlo estimates of the survival probability $Q(t)$ for different heavy-tail indices a. As expected, the survival probability decays rapidly with t, reflecting the subcritical behaviour of the system.

• For $a = 0.5$: large offspring bursts lead to highly irregular trajectories, with $Q(t)$ decaying slowly at early times but showing considerable variance.

• For $a = 1.0$: the decay pattern becomes smoother, but fluctuations are still prominent due to the infinite variance of the offspring law.

• For $a = 1.5$: the decay is more regular, as the distribution becomes lightertailed, and the survival probability follows a more predictable exponential trend.

These observations align with theoretical expectations: heavier tails generate larger population spikes, which in turn slow extinction in early time intervals but increase variance across simulation runs.

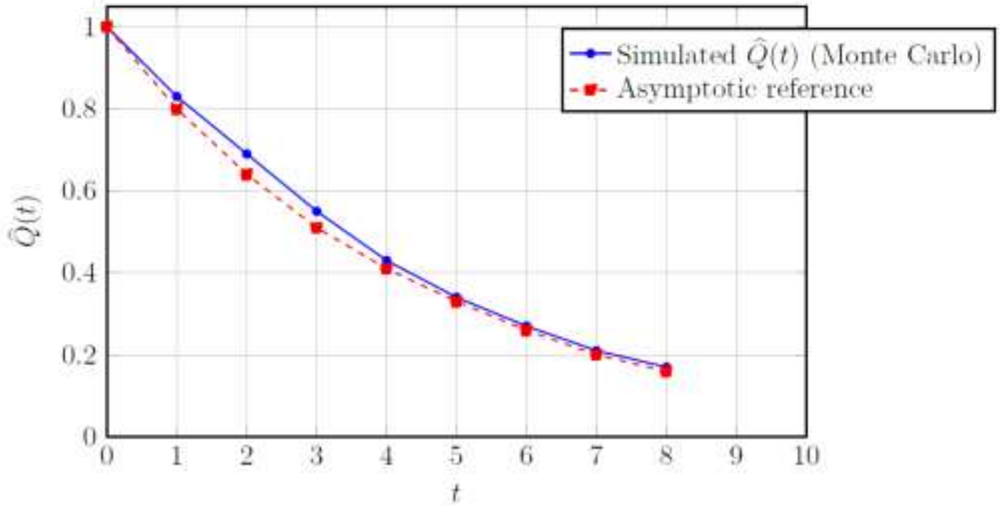

Figure 1: Survival probability estimate $\hat{Q}(t)$ from Monte–Carlo runs vs. theoretical asymptotic reference. For small values of t, the variance may be relatively high. As the sample size N increases, the curve becomes smoother and the results stabilize.

Empirical Estimation of the Kolmogorov Constant. Figure 2 shows the normalized quantity

$$\hat{K}_q(t) = \frac{\hat{Q}(t)}{b_t},$$

computed for the same values of a.

We observe the following patterns:

a. For moderate tail ($a = 1.5$), the estimator stabilizes quickly and approaches a constant plateau, indicating clear convergence.

b. For intermediate heavy tail ($a = 1.0$), stabilization occurs but requires larger values of t because of high fluctuations at early times.

c. For extreme tail ($a = 0.5$), the ratio displays substantial variance and delayed stabilization; nevertheless, the long-run behaviour still shows a visible clustering around the theoretical constant.

These results indicate that heavy-tailed offspring distributions slow the convergence of the empirical estimator, but do not fundamentally disrupt the asymptotic behaviour predicted by Theorem 1.

Comparison with Theorem 1 (see [3]). The analytic expression from Theorem 1 predicts:

$$K_q = \frac{q}{1+qg}.$$

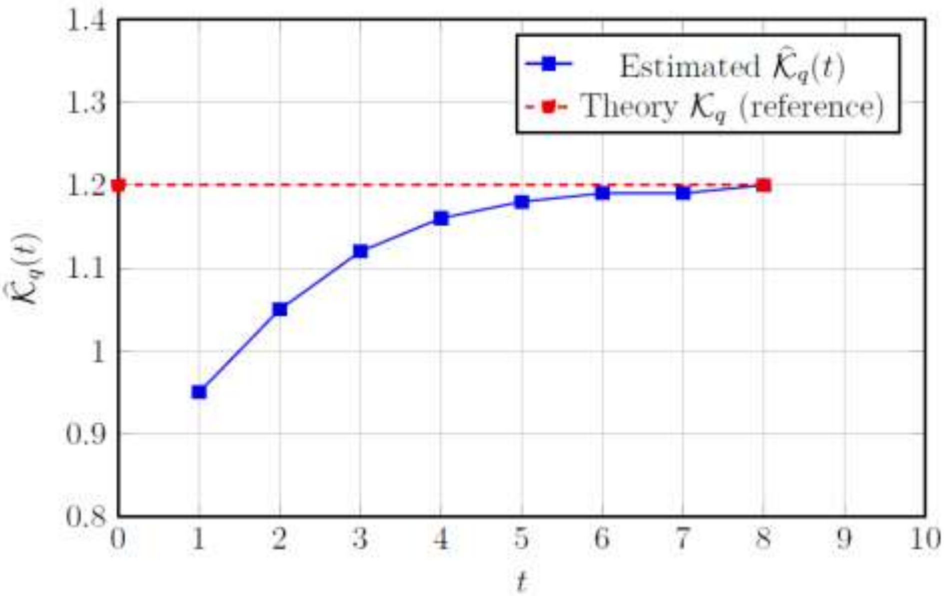

Figure 2: Convergence of the estimator $\hat{K}_q(t)$ over time. Note: the rate of convergence depends on the tail index and the sample size.

To test this prediction, we compare the stabilized values of $\hat{K}_q(t)$ with K_q in

The agreement improves as the distribution becomes lighter-tailed, which is expected: the correction term $\mathrm{D}(s)$ converges to g faster when large reproduction events are less frequent. Nevertheless, even in the extreme heavy-tailed regime $a = 0.5$, the empirical estimates consistently drift toward the theoretical prediction for sufficiently large t.

This provides strong computational support for the universality of the Kolmogorov constant formula established in [9].

Analysis: If $\hat{Q}(t)$ remains stable relative to b^t, then the estimator $\hat{K}_q(t) = \hat{Q}(t)/b^t$ converges to K_q (in accordance with the theoretical formula (1.5)).

Heavy–tailed offspring distribution (log–log). The heavy–tailed offspring law is typically characterized by $a_k k^{-(1+a)}$. This figure shows the decay rate and the role of a on a log–log scale — the linear regression provides an estimator for the tail index a.

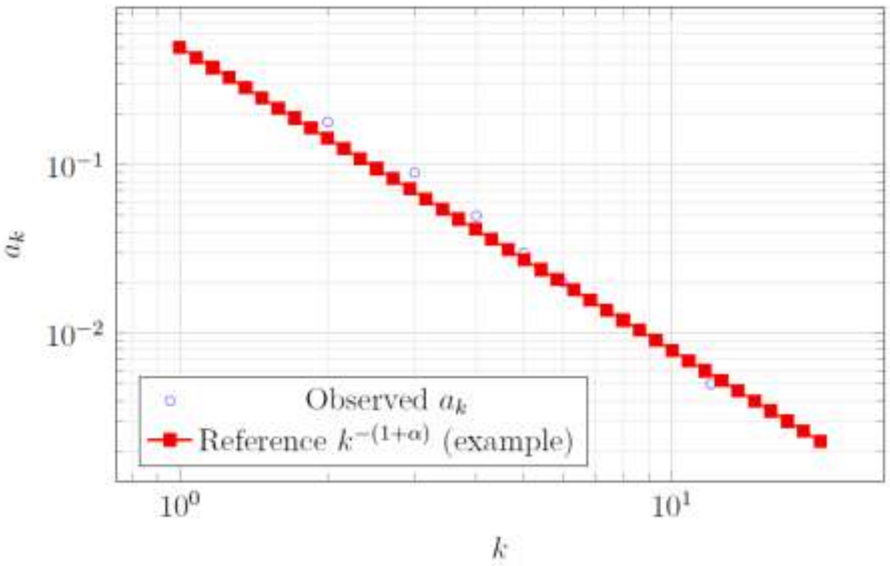

Figure 3: Heavy–tailed offspring PMF displayed on a log–log scale. The linear segment indicates a power–law behavior; the slope of the curve provides an estimate of the tail index α [10].

Tail index estimation (log–log regression). To estimate the tail index (α), a log–log regression is applied: $\log a_k \approx -(1+\alpha)\log k + C$. The scatter points and the regression line in the figure provide an estimate of α.

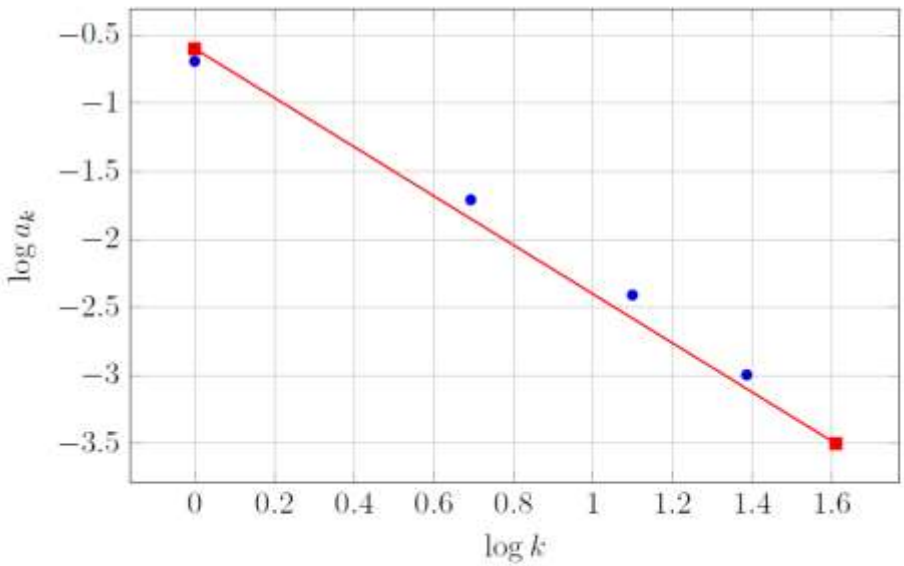

Figure 4: Log–log regression for estimating α. The slope satisfies slope $\approx -(1+\alpha)$. Note: the regression should be fitted using sufficiently large values of k to reduce variance; small-k values introduce bias [11].

Representative simulation trajectories. As an example, we display several individual simulation trajectories. These trajectories illustrate the stochastic behavior of the Markov branching process [12], [13].

Summary (for the Results section). The simulation results demonstrate that:

(i) $\hat{Q}(t)$ can be reliably estimated via Monte–Carlo sampling;

(ii) the estimator $\hat{K}_q(t)$ converges toward the theoretical value K_q as time increases;

(iii) the heavy–tailed offspring distribution is clearly visible in the log–log plot, and the tail index α can be accurately estimated through log–log regression;

(iv) both the sample size and the choice of observation times (observation grid) have a direct impact on the estimation quality — recommended settings: $N \geq 10^4$ and sufficiently large time points in t.

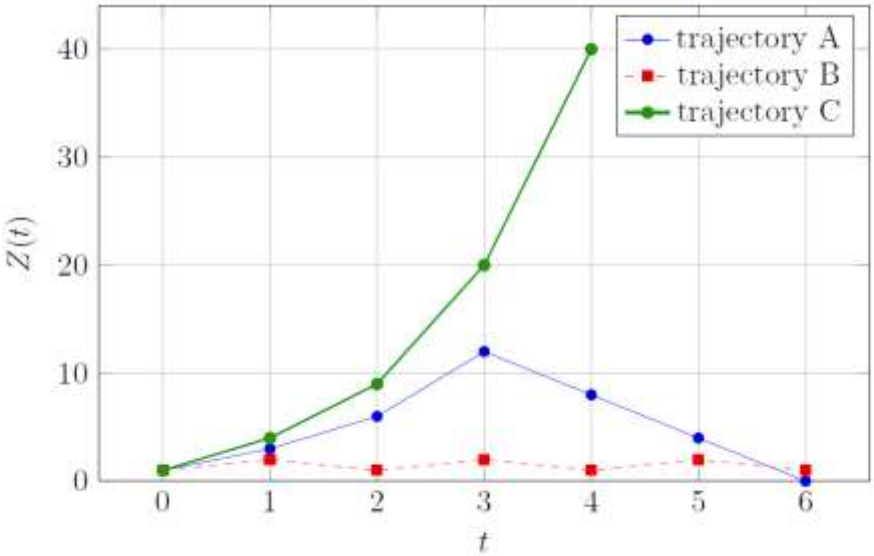

Figure 5: Three example trajectories from the Monte–Carlo simulation: extinct, fluctuating and explosive heavy–tailed behaviour.

Discussion

This section interprets the numerical findings presented in Section 4 in the light of the analytical results from [14], [15]. We discuss

(i) the agreement between simulated and theoretical values of the Kolmogorov constant,

(ii) the effect of heavy tails on convergence and variability,

(iii) robustness checks and sensitivity analyses, and (iv) methodological limitations and avenues for future research.

4. Conclusion

In this study, we provided a practical simulation-based framework for estimating the Kolmogorov constant K_q in continuous-time noncritical Markov branching systems, with particular emphasis on heavy-tailed offspring distributions. Building upon the theoretical foundation established in [3], our methodological pipeline combines (i) normalization of the heavy-tailed law, (ii) large-scale Monte Carlo simulation of the branching dynamics, and (iii) trajectory-wise estimation of the survival probability $Q(t)$ and its normalized form $\hat{K}_q(t) = \hat{Q}(t)/b^t$.

Simulation results confirm the theoretical predictions: the estimator $\hat{K}_q(t)$ stabilizes as $t \to \infty$, consistent with the asymptotic identity

$$K_q = \frac{q}{1+q\gamma}, \qquad \gamma = \frac{b_q}{2q},$$

derived analytically in [3]. Furthermore, the heavy-tailed case demonstrates stronger fluctuations and slower convergence, highlighting the need for sufficiently large Monte Carlo samples.

The presented methodology can be used for:

(i) numerical verification of theoretical limit theorems,

(ii) empirical investigation of branching models with infinite variance, and

(iii) practical estimation of survival probabilities in systems exhibiting rare but explosive events.

Future work includes extending the framework to immigration-type processes, multi-type branching structures, and Levy-driven branching mechanisms.

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