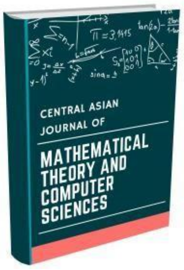

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Article

New Conditions Establish the Relationships Between Submodules

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Abstract: This paper investigates new conditions that establish bidirectional relationships among several classes of submodules within the framework of commutative rings with identity and unitary modules. The study focuses on analyzing the structural connections between prime, nearly prime, approximately prime, and almost approximately nearly prime submodules by formulating precise criteria under which these concepts become equivalent. A series of theoretical results is developed to determine when the implications between these classes can be reversed, emphasizing the influence of module-theoretic properties such as the Jacobson radical, socle, essential submodules, and semisimplicity. Various propositions are proved to demonstrate that additional structural constraints on modules or submodules lead to equivalence between generalized notions and classical primeness. Illustrative examples are provided to clarify situations in which these relationships fail, highlighting the necessity of the imposed conditions. The obtained results contribute to a deeper understanding of generalized primeness in module theory and provide a unified perspective that connects several previously introduced generalizations through clearly defined algebraic conditions.

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1. Introduction

In this search, every module is unitary and every ring is commutative. $У < D$, $У$ is named a primary if whenever $aḳ \in У$, for $a \in R$, $ḳ \in D$ implies that either $ḳ \in У$ or $a \in \sqrt{[У :_R D]}$ [1], [2]. Recently primary submodules have been generalized to Nearly and approximately primary submodule see [3]. The concept Restrict Nearly primary submodule was introduced in [4], [5], [6] as new generalized of primary submodule Where $У < D$, $У$ is named Restrict Nearly Primary submodule (simply RNPr), if whenever $aḳ \in У$, for $a \in \mathcal{R}$, $ḳ \in D$ it means that $ḳ \in У + (Soc(D) \cap J(D))$ or $a \in \sqrt{\left[\left(У + \left(Soc(D) \cap J(D)\right)\right) :_{\mathcal{R}} D\right]}$, that is $a^n D \subseteq У + (Soc(D) \cap J(D))$, $n \in Z^+$. And an ideal $\mathcal{J}$ of a ring $\mathcal{R}$ is named a RNPr ideal of $\mathcal{R}$, iff $\mathcal{J}$ is a RNPr submodule of an $\mathcal{R}$-module $\mathcal{R}$, where $J(D)$ is the Jacobson radical of D defined to be the intersection of all maximal submodule of D (if D has no maximal submodules then $J(D)$=D) or the sum of all small submodule of D [7], [8], [9]. $Soc(D)$ is the Socal of D defined to be the intersection of all essential submodule of D[10], [11], where a non-zero submodule $У$ of D is named an essential if $У \cap \beta \neq 0$ for each non-zero submodule β of D [12], [13]. The work also relied on some theoretical research such as [14], [15].

2. Materials and Methods

This study adopts the deductive approach within the framework of commutative ring theory with identity and unitary modules. The work is primarily based on theoretical investigation and algebraic analysis rather than computational or experimental methods.

The methodology is grounded in the use of fundamental definitions and well-established concepts in module theory, including commutative rings, modules, Jacobson radical, socle, essential submodules, and semisimple modules. These concepts serve as the basic structure upon which the results of this research are developed.

The study follows the following methodological steps:

1. **Theoretical Framework Construction**

   The research builds upon existing definitions and generalizations of submodules such as prime, nearly prime, approximately prime, and almost approximately nearly prime submodules, as well as recently introduced extensions in the literature.

2. **Development of New Structural Conditions**

   New algebraic conditions are introduced to establish relationships and equivalence between different generalized notions of primeness. These conditions involve structural properties of modules such as containment relations involving the Jacobson radical and the socle, as well as properties like essentiality and semisimplicity.

3. **Propositional and Logical Proof Technique**

   The results are derived through formal mathematical propositions supported by rigorous proofs. These proofs rely on standard techniques in abstract algebra, including inclusion properties, module homomorphism arguments, and radical theory.

4. **Use of Counterexamples**

   Specific examples, particularly based on finite cyclic modules such as Zn\mathbb{Z}_nZn, are constructed to demonstrate that certain implications do not hold in general. These counterexamples justify the necessity of the imposed conditions in the main results.

5. **Comparative Analysis of Generalized Concepts**

   A systematic comparison is conducted between different generalized forms of primeness in submodules in order to determine conditions under which these notions become equivalent or remain distinct.

Overall, the methodology combines structural algebraic analysis with logical reasoning to unify several generalized concepts of primeness under clearly defined algebraic conditions, thereby contributing to a deeper understanding of submodule theory in commutative algebra.

3. Results and Discussion

1. The connections between AL.App. ne. prime submodules and other related concepts.

This section presented the connections between prime, ne. prime, APP. prime and Al.APP. ne. prime submodules.

2. Prime and Al.App. nearly. Prime Submodules.

We start by first part of this section which established the relationship of prime submodules and Al.App. ne.prime submodules.

As we gave in the first section of this chapter, every prime submodule of an U-module Q is an Al.App. ne. prime submodule of Q, but contrariwise isn't true the following example explains that.

Example (3.1)

Let $Q = Z_{72}$, $U = Z$, $h = \langle\bar{8}\rangle$ is an Al.App. ne. prime of Z_{72}. $\forall\, a \in Z, b \in Z_{72}$, if $ab \in h$, $\Longrightarrow$

$ab \in h + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{8}\rangle + (\langle\overline{12}\rangle + \langle\bar{6}\rangle) = \langle\bar{2}\rangle$ or $a \in \left[h + \left(soc(Z_{72}) + J(Z_{72})\right):_Z Z_{72}\right] = [\langle\bar{2}\rangle:_Z Z_{72}] = 2Z$. That is if $4.\bar{2} = \bar{8} \in h$, for $4 \in Z$, $\bar{2} \in Z_{72}$, that means $\bar{2} \in h + \left(soc(Z_{72}) + J(Z_{72})\right) = \langle\bar{2}\rangle$ and $4 \in \left[h + \left(soc(Z_{72}) + J(Z_{72})\right):_Z Z_{72}\right] = 2Z$. But $\langle\bar{8}\rangle$ is not prime of Z_{72}.

The following findings demonstrated that, in some circumstances, the opposite implication is true.

Proposition(3.2)

If Q be an U-module, h is proper of Q with $J(Q) \subseteq h$, $soc(Q) \subseteq h$. Then h is prime iff h is Al.App. ne. prime.

Proof

$(\Leftarrow)$ $J(Q) \subseteq h$, if $ab \in h$ for $a \in U$, $b \in Q$. h is Al.App. ne.prime, then $b \in h + (soc(Q) + J(Q))$ or $aQ \subseteq h + (soc(Q) + J(Q))$. But $soc(Q) \subseteq h$ and $J(Q) \subseteq h \Longrightarrow h + soc(Q) = h$, $h + (soc(Q) + J(Q)) = h + J(Q) = h$. Thus $b \in h$ or $aQ \subseteq h$. Therefore h is prime of Q.

$(\Longrightarrow)$ Direct.

Proposition(3.3)

If Q be an U-module with $J(Q) \subseteq h, soc(Q) = 0$. Then h is prime iff h is Al.App. ne.prime of Q.

Proof

$(\Longleftarrow)$ If $ab \in h$ for $a \in U$, $b \in Q$, hence $b \in h + soc(Q) + J(Q)$ or $aQ \subseteq h + (soc(Q) + J(Q))$. Now $soc(Q) = 0$, then $soc(Q) \subseteq h$ and we have $J(Q) \subseteq h \Longrightarrow h + soc(Q) = h$ and $h + J(Q) = h$, thus $h + (soc(Q) + J(Q)) = h$. Hence $b \in h$ or $aQ \subseteq h$.

$(\Longrightarrow)$ Direct.

Proposition(3.4)

If Q is a semi.simple with $soc(Q) = (0)$. Then h is prime iff h is Al.App. ne.prime of Q.

Proof

$(\Longleftarrow)$ If $ab \in h$ for $a \in U$, $b \in Q$, hence $b \in h + soc(Q) + J(Q)$ or $aQ \subseteq h + (soc(Q) + J(Q))$. Now $soc(Q) = 0$, then $soc(Q) \subseteq h$ and we have Q is a semi.simple $\Longrightarrow J(Q) = 0 \subseteq h \Longrightarrow h + soc(Q) = h$ and $h + J(Q) = h$, thus $h + (soc(Q) + J(Q)) = h$. Hence $b \in h$ or $aQ \subseteq h$.

$(\Longrightarrow)$ Direct.

3. Ne.Prime and Al. App. ne. Prime Submodules.

The relationship between ne. prime submodules was established in the second section and Al.App. ne.prime submodules.

As we gave in the first section, every ne. prime is an Al.App. ne.prime submodule of Q, but contrariwise isn't true This is explained in the example that follows.

Example(4.1)

If $Q = Z_{12}$, $U = Z$, $h = \langle\bar{6}\rangle$ is an Al.App. ne. prime of Z_{12}. Thus $a \in Z, b \in Z_{12}$, if $ab \in h \Longrightarrow b \in h + \left(soc(Z_{12}) + J(Z_{12})\right) = \langle\bar{6}\rangle + (\langle\bar{2}\rangle + \langle\bar{6}\rangle) = \langle\bar{6}\rangle + \langle\bar{2}\rangle = \langle\bar{2}\rangle$ or $a \in \left[h + \left(soc(Z_{12}) + J(Z_{12})\right):_Z Z_{12}\right] = [\langle\bar{2}\rangle:_Z Z_{12}] = 2Z$. if $3.\bar{2} \in h$, for $3 \in Z$, $\bar{2} \in Z_{12} \Longrightarrow \bar{2} \in h + \left(soc(Z_{12}) + J(Z_{12})\right) = \langle\bar{2}\rangle$. But h is not ne. prime of Z_{12} , because $3.\bar{2} \in h$, but $\bar{2} \notin h + J(Z_{12}) = \langle\bar{6}\rangle + \langle\bar{6}\rangle = \langle\bar{6}\rangle$ and $3 \notin [h + J(Z_{12}):_Z Z_{12}] = [\langle\bar{6}\rangle:_Z Z_{12}] = 6Z$.

The following findings demonstrated that, in some circumstances, the opposite implication is true.

Proposition(4.2)

If h be a proper submodule with h is essential of an U-module $Q, J(Q) \subseteq h$. Then h is ne. prime iff h is Al.App. ne. prime of Q.

Proof

($\Leftarrow$) If $ab \in h$ for $a \in U, b \in Q$. h is Al.App. ne. prime, then $b \in h + soc(Q) + J(Q)$ or $aQ \subseteq h + soc(Q) + J(Q)$. We have h is essential then $soc(Q) \subseteq h \Longrightarrow soc(Q) + h = h$, $J(Q) \subseteq h$ thus $b \in h + J(Q)$ or $aQ \subseteq h + J(Q)$. Hence h is ne. prime of Q.

($\Longrightarrow$) Direct.

Proposition(4.3)

If h contains every simple submodule of Q. Then h is ne. prime iff h is Al.App. ne. prime of Q.

Proof

($\Leftarrow$) If $ab \in h$ for $a \in U, b \in Q$. h is Al.App. ne. prime, then $b \in h + (soc(Q) + J(Q))$ or $aQ \subseteq h + (soc(Q) + J(Q))$. Now h contains every simple submodule of Q $\Longrightarrow soc(Q) \subseteq h$, then $h + soc(Q) = h$, so $h + (soc(Q) + J(Q)) = h + J(Q)$. Thus $b \in h + J(Q)$ or $aQ \subseteq h + J(Q)$. Hence h is ne. prime of Q.

($\Longrightarrow$) Direct.

The proofs of the following results are direct.

4. App. Prime and Almost App. ne. Prime Submodules.

As we mentioned in the upper part of this research, every App. prime is an Al.App. ne. prime of Q, but contrariwise isn't true the following example explains that.

Example(5.1)

If $Q = Z_{72}$, $U = Z$, $h = \langle\bar{4}\rangle$ is an Al.App. ne. prime of Z_{72}. But h is not App. prime of Z_{72}, because $2.\bar{2} \in h$, but $\bar{2} \notin h + soc(Z_{72}) = \langle\bar{4}\rangle + \langle\overline{12}\rangle = \langle\bar{4}\rangle$ and $2 \notin [h + soc(Z_{72}):_Z Z_{72}] = [\langle\bar{4}\rangle:_Z Z_{72}] = 4Z$.

The following findings demonstrated that, in some circumstances, the opposite implication is true.

Proposition(5.2)

If Q is a semi.simple. Then h is App prime iff h is Al.App. ne. prime of Q.

Proof

($\Leftarrow$) If $ab \in h$ for $a \in U, b \in Q$. h is Al.App. ne. prime, then $b \in h + (soc(Q) + J(Q))$ or $aQ \subseteq h + (soc(Q) + J(Q))$. Since Q is a semi.simple then $J(Q) = 0 \subseteq h$, then $h + J(Q) = h$, so $h + (soc(Q) + J(Q)) = h + soc(Q)$. Thus $b \in h + soc(Q)$ or $aQ \subseteq h + soc(Q)$. Hence h is App. prime of Q.

($\Longrightarrow$) Direct.

Proposition(5.3)

If Q has no maximal submodule and h is a proper submodule of Q. Then h is App. prime iff h is Al.App. ne. prime of Q.

Proof

($\Leftarrow$) Since Q has no maximal $\Longrightarrow J(Q) \subseteq soc(Q)$, then $soc(Q) + J(Q) = soc(Q)$, so $h + (soc(Q) + J(Q)) = h + soc(Q)$, if $ab \in h$ for $a \in U, b \in Q$. h is Al.App. ne. prime, then $b \in h + (soc(Q) + J(Q)) = h + soc(Q)$ or $aQ \subseteq h + (soc(Q) + J(Q)) = h + soc(Q)$. Thus $b \in h + soc(Q)$ or $aQ \subseteq h + soc(Q)$. Hence h is App. prime of Q.

($\Longrightarrow$) Direct.

4. Conclusionc

The present study developed a systematic framework for understanding the relationships among several generalized notions of primeness in module theory. By introducing suitable structural conditions on modules and submodules, clear criteria were established under which prime, nearly prime, approximately prime, and almost approximately nearly prime submodules become logically equivalent. The results demonstrate that the behavior of these generalized concepts is strongly influenced by intrinsic module properties such as the Jacobson radical, the socle, essential submodules, and semisimple structure. The obtained propositions showed that although generalized primeness concepts do not coincide in general, imposing additional algebraic restrictions enables the reversal of implications that normally hold in only one direction. Constructed examples confirmed the independence between these classes in unrestricted cases and emphasized the necessity of the stated assumptions. Furthermore, the study clarified how containment relations involving J(Q) and Soc(Q) play a central role in reducing generalized conditions to classical primeness. Overall, the research provides a unified perspective that connects different extensions of prime submodules through precise algebraic conditions, thereby enriching the theoretical understanding of submodule structures. These findings open pathways for further investigations into broader generalizations of primeness and their applications within module and ring theory.

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