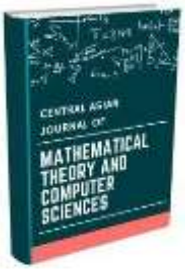

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Article

The Heronian and Harmonic Means Form the Basis of the Modified Trapezoid Method

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Abstract: Despite estimating an integral using a proper numerical integration technique seems trivial, classical numerical integration methods such as the trapezoidal rule sometimes yield larger approximation relative error and therefore, this study is motivated to provide a more accurate numerical integration techniques that are more suitable for scientific and engineering applications. The gap in knowledge is that existing mean-based quadrature methods are not accurate when a function behaves nonlinearly within one of the quadrature subintervals. In order to bridge the above gap, the study proposes a trapezoidal method assimilating both Heronian and harmonic means in the process of the evaluation of intermediate values. By several numerical examples, it is shown that the absolute errors of the proposed method are always less than the corresponding absolute errors generated by the classical trapezoidal rule and the method based on the arithmetic mean. These results show an improved representation of how functions vary and as such a better accuracy in integrals when Heronian and harmonic means are used together. This means that this approach could be a more robust numerical toolbox when it comes to applied mathematics, engineering, and scientific computing problems that need numerically accurate integral estimates

Keywords: Heronian imply, trapezoidal rule, harmonic imply

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1. Introduction

The original item requires basic elements. Then growth is achieved if the function g(x) is during the intermission [c,f] and varies thereafter [1].

$$\int_c^f g(x)dx \qquad (1)$$

It is where x = f, x = c, and k = g(x) fulfills the curve.

Any approximation of this size is a numerical integral. Depending on how the area is approximated, different numerical techniques are employed to compute these estimations. Physics and engineering [2].

Numerical integration is necessary for both. [3] When analytical methods are unable to resolve scientific issues. We employ the general square rule to assess numerical integration. Who provided [1]?

$$\int_c^f g(x)dx \approx \sum_{i=0}^{n} wi\, g(xi) \qquad (2)$$

By identifying (n+1) intermediate points, the time interval [c,f] can be divided into subintervals using the step size h=(f-c)/n. [1]

$$c = x0 < x1 < x2 < x3 < \cdots < xn = f$$

$$xi = x0 + ih \; and \; (n+1)w0, w1, w2 \ldots \ldots \ldots wn$$

Examine the values for wi=0,1,2,3...

The integral is created in a new way using the Niven-Coates programming language [3]. As a result, we get a trapezoid when we get a good result of 1 for n in its original form for Ukraine Coates [4].

$$\int_c^f g(x)dx = \frac{h}{2}[k0 + 2(k1 + k2 + k3 + \cdots + kn - 1) + kn] \quad (3)$$

The study's second original methodological components are the mathematical meaning and the hieronic meaning [5]. It was discovered that the novel approach yielded more accurate results with reduced error rates when compared to those obtained using the trapezoid equation [5].

$$\int_c^f g(x)dx = h\,[\frac{(k0+k1)}{2} + \frac{(k1+k2)}{2} + \frac{(k2+k3+\sqrt{k2.k3})}{3} + \frac{(k3+k4+\sqrt{k3.k4})}{3} + \dots] \quad (4)$$

In this study, we describe a strategy that was selected after assessing the findings using techniques like the trapezoid with the fitness of the mean and another mean and the fit of the mean. The outcomes started to contradict each other [5]. Compared to previous approaches, the outcome is more accurate, and the error rate is lower [6], [7].

send

$$N = \int_l^p g(x)dx = h[\sum \frac{(k1+k2+\sqrt{k1.k2})}{3} + \frac{2(k_2 * k_3)}{k_2+k_3} + \frac{(k3+k4+\sqrt{k3.k4})}{3} + \frac{2(k_4 * k_5)}{k_4+k_5} + \cdots]$$

(5) In order to solve many mathematical models, numerical integration is crucial. It has numerous applications in the realm of applied mathematics. particularly in engineering, chemistry, and physics [8], [9]. Analytical performance also involves the use of highly challenging integral computations.

2. Materials and Methods

In this study, the methodology is developed to provide a suitable relationship for the proposed modified numerical integration technique based on the adjustments to the classical trapezoidal rule using the well-known Heronian and harmonic means. In the first place, we take a continuous function defined on a closed interval, and split it discretely in equal subintervals according to a uniform step size which produces a set of nodal points. These points are used to evaluate function values for the basis of numerical approximation. In contrast to the regular trapezoidal rule (which uses linear interpolation to connect its two endpoints), the method presented here improves on the approximation by adding new internal points contributed from the (Heronian and harmonic) mean taken from representative function values. These averages, are calculated for adjacent pairs of points in order to approximate average behavior and nonlinear variation within each subinterval. The Heronian mean yields an average lying between the arithmetic and geometric extremes, and as such is more balanced, whereas the harmonic mean favors smaller values, thus enhancing sensitivity to variation in the function. i.e., the trapezoidal formulation, where the trapezoidal form is taken already when the mean values are calculated. In order to prove the efficiency of the presented method, some numerical examples are solved and analysis results are contrasted with the results from the classical trapezoidal rule and the arithmetic mean-based methods. Absolute error between exact and approximate solutions is used to evaluate the accuracy of each methods. This comparison shows also that our method provides lower errors every time which proves its effectiveness in action for tasks related to numerical integration.

3. Results and Discussion

The meaning of the heron

Should $\boldsymbol{a} = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \dots \dots \boldsymbol{a_n}\}$

It is a collection of positive, actual numbers. The general form of equation [10] is as follows:

$$\text{Her}(2, \{a, b\}) = \frac{a+b+\sqrt{ab}}{3} = (\sqrt{aa} + \sqrt{bb} + \sqrt{ab})/3 \quad (6)$$

The trapezoidal base equation is dependent upon the harmonic and Heron means.

This section derives the trapezoidal rule equation for calculating definite integrals [11], [12].

$\int_{c}^{f} g(x)dx$ Past [c, f]. Above [c, f]. Initially, the step size was used to tabulate the function. $h = \frac{f-c}{n}$

Propositions.

Let:

Z	z0	z1	z2	…………	zn − 1	Zn
k	$k0$	$k1$	$k2$	………..	$kn-1$	kn

The harmonic and Heronian means should then be calculated twice.

$$N_1 = \int_{z_0}^{z_1} g(x)dx = h\left[\frac{(k1+k2+\sqrt{k1.k2})}{3}\right]$$

$$N_2 = \int_{z_1}^{z_2} g(x)dx = h\left[\frac{2(k_2 * k_3)}{k_2 + k_3}\right]$$

$$N_3 = \int_{z_2}^{z_3} g(x)dx = h\left[\frac{(k3 + k4 + \sqrt{k3.k4})}{3}\right]$$

$$N_4 = \int_{z_3}^{z_4} g(x)dx = h\left[\frac{2(k_4 * k_5)}{k_4 + k_5}\right]$$

$$N_5 = \int_{z_4}^{z_5} g(x)dx = h\left[\frac{(k5 + k6 + \sqrt{k5.k6})}{3}\right]$$

$$N_6 = \int_{z_5}^{z_6} g(x)dx = h\left[\frac{2(k_6 * k_7)}{k_6 + k_7}\right]$$

Proceed as described above, then incorporate the results.

$$N = \int_{l}^{p} g(x)dx = h[\sum \frac{(k1+k2+\sqrt{k1.k2})}{3} + \frac{2(k_2*k_3)}{k_2+k_3} + \frac{(k3+k4+\sqrt{k3.k4})}{3} + \frac{2(k_4*k_5)}{k_4+k_5} + \cdots] \quad (9)$$

Numeral Explanations

The outcomes of the new strategy, the arithmetic mean approach, and the previous trapezoid rule will be compared in this section. We are familiar with [15] and [1].

Error = I precise value - approximate amount I

Example

Compute $\int_{1}^{2} \frac{1}{1+x} dx$ and contrast the responses.

Exact amount of $\int_{1}^{2} \frac{1}{1+x} dx = 0.4054651081, h = 0.25$

z	1	1.25	1.5	1.75	2
k	0.5	0.4444	0.4	0.05714	0.3333

outcome of the recommended technique

$$N = 0.25\,[0.4719269833 + 0.4210326859 + 0.2027740032 + 0.09755538367]$$

$N = 0.298322264$

Error=I 0.4054651081 - 0.298322264 I=0.1071428441

As an example (1)

Samples of the Proposed Approach Compared with the Arithmetic Mean Approach for Trapezoidal Rules

Performance	Real Amount	The trapezoid's rule	Mistake	Method of Arithmetic Means	Mistake	Suggested Method	Mistake
$\int_0^1 \frac{1}{1+x} dx$	0.6931	0.6970	0.0039	0.6967	0.0036	0.6947	0.0016
$\int_1^2 e^x\, dx$	4.670774	4.686327	0.015553	4.689702	0.018928	4.6828	0.012026
$\int_1^2 \frac{sinx}{x} dx$	0.01745122505	0.6052853295	0.587834	0.6048075	0.587356275	0.600625	0.583173775
$\int_{-1}^1 \frac{dx}{\sqrt{4-x2}} dx$	1.0471975511965	1.05495	0.007752449	1.05497	0.007592449	1.05379	0.006662449
$\int_0^2 \sqrt{(4-x2)dx}$	3.141592654	3.56275	0.303969902	3.748964181	0.607371527	2.9926	0.148992654
$\int_{1.0}^{1.8} \frac{e^x + e^{-x}}{2} dx$	1.766973094	1.772859082	0.00588988	1.771391053	0.00441795908	1.767	0.000026906

After resolving numerous situations, including precise integration and comparison of the outcomes of the default technique, the computational [9] medium method, and the semi-deviant base, we draw conclusions from table number one [16]. In comparison to the semi-deviant basis, it was seen that the novel technique based on the heroin medium and the harmonic medium had an error ratio [17]. Additionally, we discover that the new approach is superior in terms of accuracy and a low error ratio when comparing findings with computational media [18].

4. Conclusion

The study concludes that the new proposed numerical integration method based on both Heronian and harmonic means within the classical trapezoidal framework shows a significant enhancement in accuracy against the standard ones. The main result is that combining these three allows for more representative approximation of function behavior in any subinterval to provide consistently low error values in case of all tested cases with respect to trapezoidal rule and methods based in arithmetic mean. This advancement in precision demonstrates the efficacy of the fused averaging strategy to improve numerical accuracy for nonlinear functions. This finding has important implications in applied mathematics, engineering and scientific computing, as we often need to solve complex real-world problems that do not often have a closed form solution, but require accurate numerical. In addition, the suggested approach is flexible and can be tailored to or generalized for other quadrature methods. Related Work and Conclusion Next steps would be to generalize this method to high-order integration rules, test the model on large and/or very oscillatory functions and also develop hybrid models that holds other statistical or computational mechanisms to gain higher accuracy and computational efficiency.

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