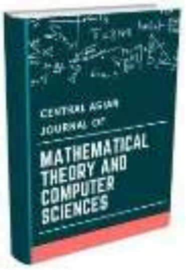

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Article

Harmonic Structures of Hilbert Spaces over Compact Abelian Groups

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Abstract: This paper investigates harmonic Hilbert spaces defined over compact abelian groups from a functional analytic and harmonic analysis perspective. These spaces are constructed via Fourier analytic techniques on the dual group, incorporating suitable weighting functions to control regularity and algebraic behavior. We examine conditions under which such Hilbert spaces admit a reproducing kernel structure and remain stable under pointwise multiplication of functions. Special attention is given to identifying sufficient assumptions on the associated weight functions that guarantee the resulting spaces form Banach algebras equipped with a natural involution induced by complex conjugation. Under these assumptions, the studied spaces exhibit symmetry properties that allow them to be treated as invaolutive Banach algebras within a harmonic framework. Spectral characteristics of these algebras are also analyzed, and criteria are provided ensuring that their spectra coincide with those of the algebra of continuous functions on the underlying compact group. Furthermore, the paper explores parametric families of harmonic Hilbert spaces generated by semigroups of self-adjoint operators acting on L^2 (G). Within this setting, a close relationship is established between convolution-type bounds on the defining functions and the existence of a compatible algebraic structure. Finally, connections between the proposed harmonic Hilbert spaces and certain Fourier-type function spaces are discussed, highlighting their relevance to approximation theory and harmonic analysis on compact groups.

Keywords: Compact abelian groups; Harmonic analysis; Reproducing kernel Hilbert spaces; Banach algebras; Fourier analysis

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1. Introduction

Harmonic analysis provides a natural framework for studying function spaces defined on locally compact abelian groups through their spectral representations. One important class of such spaces arises from Hilbert spaces constructed via Fourier transforms of weighted square-integrable functions on the dual group. These spaces have attracted considerable attention due to their rich algebraic structure and their applications in approximation theory and operator analysis.

Let G be a compact abelian group with dual group $\hat{G}$. Given a measurable weight function $\mu:\hat{G}\to(0,\infty)$ satisfying suitable integrability conditions, we consider the Hilbert space

$$H_\mu:=F^{-1}\left(L^2(\hat{G},\mu(\gamma)\,d\gamma)\right),$$

where F denotes the Fourier transform on G. Spaces of this form consist of continuous functions on G and naturally admit a reproducing kernel structure, making them examples of reproducing kernel Hilbert spaces (RKHSs). Similar constructions have appeared in various contexts of harmonic and functional analysis; see, for example, [1, 2, 3].

A fundamental question concerns the stability of H_μ under pointwise multiplication. This property is closely related to growth and regularity conditions imposed on the weight μ. One commonly studied assumption is a moderated growth condition of the form

$$\mu(\gamma_1+\gamma_2)\le C\,\mu(\gamma_1)\mu(\gamma_2), \qquad \gamma_1,\gamma_2\in \hat{G}, \qquad (1)$$

Assume the existence of a positive constant $C>0$. Under this assumption, it can be established that the space H_μ is closed under pointwise multiplication and, moreover, admits the structure of a Banach algebra. Related analytical foundations and proofs of this property have been discussed in earlier works [4,5].

The present study is devoted to the investigation of harmonic Hilbert spaces associated with weight functions $\mu\in L^1(\hat{G})$ that fulfill an inequality of convolution type, which serves as a fundamental condition for the analysis carried out herein.

$$(\mu*\mu)(\gamma)\le C\,\mu(\gamma), \qquad \gamma\in \hat{G}, \qquad (2)$$

together with symmetry and positivity assumptions. Conditions of this type naturally arise in the study of convolution algebras and have been investigated in related settings in [6, 7]. A typical example on the dual group $\hat{G}\cong Z^d$ of the torus $G=T^d$ is given by polynomially weighted functions

$$\mu(\gamma)=1/((1+|\gamma|^2)^\alpha), \qquad \alpha>d/2. \quad (3)$$

A central result of this article establishes that, when the underlying group is compact and abelian, the associated function spaces H_μ possess a symmetric Banach *-algebra structure. This structure arises naturally from the closure of H_μ under pointwise products together with the involution induced by complex conjugation. In addition, it is shown that the maximal ideal space of H_μ is identical to that of the C^*-algebra $C(G)$ consisting of continuous functions defined on the group G. This result reveals a deep correspondence between harmonic Hilbert-type constructions and classical algebras of continuous functions.

The analysis is then extended to collections of reproducing kernel Hilbert spaces obtained through strongly continuous semigroups of self-adjoint operators acting on $L^2(G)$. Within this framework, we demonstrate that a specific convolution-based constraint provides a necessary and sufficient criterion for these spaces to support a Banach algebra structure compatible with their Hilbert space geometry.

Finally, we examine inclusion relationships between the newly introduced harmonic Hilbert spaces and certain weighted Fourier-style algebras defined on compact abelian groups. These relationships emphasize the applicability of the proposed framework to problems in spectral theory as well as approximation tasks in high-dimensional settings.

Related Work

Weighted Banach algebras on locally compact abelian groups have been studied extensively due to their connections with harmonic analysis, convolution operators, and spectral theory. Early foundational work established that convolution algebras defined by weighted function spaces are closely linked to structural properties of the underlying weight functions. In particular, it was shown that certain convolution algebras attain completeness precisely when the associated weights satisfy convolution bounds; see, for instance, [8].

One important class of weight conditions arises from convolution inequalities on the dual group. Given a weight $w:\hat{G}\to(0,\infty)$, the space L_w^∞ forms a Banach convolution algebra if and only if the inverse weight w^{-1} satisfies a convolution dominance condition of the form

$$(w^{-1}*w^{-1})(\gamma)\le C\,w^{-1}(\gamma), \qquad \gamma\in \hat{G}, \qquad (4)$$

for some constant $C>0$. It is also known that subadditive weights whose inverses are integrable automatically satisfy such inequalities, providing a practical method for constructing convolution algebras; however, subadditivity and subconvolutivity are, in general, independent properties [9].

Another well-studied condition is submultiplicativity, expressed as

$$w(\gamma_1+\gamma_2)\le C\,w(\gamma_1)w(\gamma_2), \qquad \gamma_1,\gamma_2\in \hat{G}, \qquad (5)$$

which characterizes when the weighted space L_w^1 becomes a Banach convolution algebra. This result has played a central role in the theory of weighted group algebras and their applications; see [10].

Beyond the cases $p=1$ and $p=\infty$, weighted L^p convolution algebras for $1<p<\infty$ have also been investigated. Sufficient conditions ensuring algebra structure typically involve convolution estimates on powers of the weight function. For instance, it has been shown that L_w^p is closed under convolution provided

$$(w^{-p} * w^{-p})(\gamma) \le C\, w^{-p}(\gamma), \qquad \gamma \in \hat{G}, \qquad (6)$$

a condition that reduces, in the Hilbert space case $p=2$, to a subconvolutivity assumption compatible with the construction of harmonic Hilbert spaces [11]. Related developments for Euclidean groups and polynomial weights can be found in [12].

The spectral theory of weighted convolution algebras has also received considerable attention. Results concerning semisimplicity, maximal ideal spaces, and spectral invariance have been obtained for various classes of weights on abelian groups [13, 14]. More recently, representation-theoretic aspects of convolution Banach *-algebras, including symmetry properties of their spectra, have been studied in both abelian and non-abelian settings [15].

A different but related approach combines Banach algebra structures with Hilbert space geometry. In this direction, Ambrose introduced the notion of H^*-algebras, which satisfy the identity

$$\langle fg,h\rangle = \langle g, f^* h\rangle, \qquad f,g,h \in H, \qquad (7)$$

implying that multiplication operators act as isometries on the Hilbert space [16]. Classical examples include L^2 convolution algebras on compact groups. However, harmonic Hilbert spaces and reproducing kernel Hilbert algebras considered in the present work do not generally fulfill this strong compatibility condition.

The focus of this paper is restricted to compact abelian groups, where Fourier series expansions and kernel methods allow for a precise spectral analysis. Within this framework, the coefficients that determine the construction can be interpreted as the spectral values associated with compact integral operators acting on the space $L^2(G)$ and commuting with group translations. Moreover, the irreducible characters of the group constitute an orthonormal basis that diagonalizes these operators. Such a spectral decomposition allows the application of kernel representation theorems of the Mercer class, which in turn provide a rigorous description of both the algebraic structure and the spectral behavior of the induced reproducing kernel Hilbert algebras.

Plan of the Paper

The exposition is structured in a progressive manner. We begin in Section 2 by fixing the notation and recalling foundational notions from harmonic analysis in the context of compact abelian groups. Section 3 is dedicated to summarizing the core principles of reproducing kernel Hilbert space theory that form the analytical backbone of this work. The subsequent section presents a systematic construction of reproducing kernel Hilbert algebras arising from subconvolutive functions defined on the dual group. A detailed investigation of the spectral characteristics and associated state spaces of these algebras is carried out in Section 5. In Section 6, we focus on continuous one-parameter collections of reproducing kernel Hilbert algebras induced by Markov semigroups. Section 7 addresses weighted Fourier-type algebras and proves embedding relationships linking them to the harmonic Hilbert spaces developed earlier. Finally, the appendix provides a complete proof of a supplementary technical proposition concerning integral operators invariant under translation.

2. Materials and Methods

Notation and Preliminaries

Locally Compact Abelian Groups

Let G be a locally compact abelian group endowed with a Haar measure ν. Whenever G is compact, we assume that ν is normalized so that $\nu(G)=1$. The dual group of G, denoted by $\hat{G}$, consists of all continuous group homomorphisms $\chi: G \to S^1$, where S^1 denotes the unit circle in $\mathbb{C}$. The group $\hat{G}$ is equipped with its canonical dual measure $\hat{\nu}$.

Each element of $\hat{G}$ is identified with a bounded continuous function on G acting multiplicatively. The identity element of $\hat{G}$ will be denoted by $e_{\hat{G}}$. For $\chi \in \hat{G}$, its inverse character will be written as χ^{-1} whenever no ambiguity arises. If G is compact, then $\hat{G}$ is discrete and $\hat{\nu}$ reduces to a weighted counting measure [12-17].

We write C(G) for the Banach space of complex-valued continuous functions on G equipped with the uniform norm. Similarly, $C(\hat{G})$ denotes the space of continuous functions on the dual group. The Fourier transform $F:L^1(G)\to C(\hat{G})$ and the inverse Fourier transform $F^{-1}:L^1(\hat{G})\to C(G)$ are defined by

$$(Ff)(\chi):=\int_G f(x)\overline{\chi(x)}\, d\nu(x), \qquad (F^{-1}g)(x):=\int_{\hat{G}} g(\chi)\chi(x)\, d\hat{\nu}(\chi).$$

The convolution of functions $f,g\in L^1(G)$ is given by

$$(f\star g)(x):=\int_G f(y)g(x-y)\, d\nu(y),$$

and the involution is defined as $f^\sharp(x):=\overline{f(-x)}$. Analogous operations are defined on $L^1(\hat{G})$.

For Hilbert spaces such as $L^2(G)$, the inner product is assumed to be conjugate linear in the first argument, i.e.,

$$\langle f,g\rangle_{L^2(G)}:=\int_G \overline{f(x)}\, g(x)\, d\nu(x).$$

For each $x\in G$, let T_x denote the translation operator defined by $(T_x f)(y):=f(y+x)$. The family $\{T_x\}_{x\in G}$ forms a strongly continuous group of isometries on $L^p(G)$ for $1\le p<\infty$ as well as on C(G). The Fourier transform satisfies the covariance relation [12].

$$F(T_x f)(\chi)=\overline{\chi(x)}\,(Ff)(\chi), \qquad f\in L^1(G). \quad (8)$$

By standard extension arguments, F extends uniquely to a unitary operator from $L^2(G)$ onto $L^2(\hat{G})$. Moreover, for $f,g\in L^2(G)$, their convolution belongs to C(G) and admits the representation

$$(f\star g)(x)=\langle \hat{f},\widehat{(T_x g)}\rangle_{L^2(\hat{G})}. \quad (9)$$

A function $f:G\to C$ is said to be uniformly continuous if for every $\varepsilon>0$ there exists a neighborhood U of the identity in G such that $|f(x)-f(y)|<\varepsilon$ whenever $x-y\in U$.

Hilbert Algebra Formations Arising from Kernel Techniques

Assume that $\varphi:\hat{G}\to(0,+\infty)$ is a measurable function that is everywhere positive and integrable over the dual group. Based on φ, one defines a kernel $K:G\times G\to C$ that is invariant under group translations, given by an integral representation over $\hat{G}$.

$$K(x,y):=\varphi(x-y), \qquad \varphi:=F^{-1}\varrho. \quad (10)$$

Assume that ψ is a continuous function on G. By applying a classical characterization of positive definite functions on compact groups, one concludes that the kernel K is positive definite. Consequently, K determines a Hilbert space H_φ consisting of continuous functions on G, for which K plays the role of a reproducing kernel. In particular, for any $h\in H_\varphi$ and any $y\in G$, the evaluation functional is realized through the identity [13].

$$h(y)=\langle h,K(\cdot,y)\rangle_{H_\varphi}.$$

The space H_φ can be interpreted as a Hilbert space of harmonic type associated with the weighting function $\varphi^{-1/2}$ defined on the dual group. We refer to H_φ as a reproducing kernel Hilbert algebra whenever it is complete with respect to its inner product and stable under pointwise multiplication as well as involution given by complex conjugation. More explicitly, there exists a constant $\kappa>0$ such that the inequalities

$$\|uv\|_{H_\varphi}\le\kappa\, \|u\|_{H_\varphi}\, \|v\|_{H_\varphi}, \qquad \|u^*\|_{H_\varphi}=\|u\|_{H_\varphi} \quad (11)$$

hold for all $u,v\in H_\varphi$.

If H_ϱ contains a multiplicative identity, it will be denoted by 1_G. The spectrum of an element $f\in H_\varrho$, denoted $\sigma_\varrho(f)$, is defined as the set of complex numbers λ such that $f-\lambda 1_G$ is not invertible in H_ϱ. The algebra H_ϱ is called symmetric if $\sigma_\varrho(f^\sharp f)\subset[0,\infty)$ for all $f\in H_\varrho$.

Allowing a constant C in (11) does not affect the algebraic or spectral properties of H_ϱ, since an equivalent norm may be obtained by rescaling the reproducing kernel. This flexibility is useful in applications where the kernel arises naturally from integral operators or stochastic transition kernels without additional normalization. [14].

3 Outcomes Derived from the Theory of Reproducing Kernel Hilbert Spaces

This section collects a number of fundamental statements from the theory of Hilbert spaces generated by positive definite kernels, formulated in a manner suited to harmonic-analytic applications on compact abelian groups.

Let Y be a locally compact Hausdorff space. We write M(Y) for the set of all Borel probability measures on Y. Given $\mu \in M(Y)$ and a bounded measurable function g on Y, its mean value with respect to μ is defined by [15].

$$E_\mu(g) := \int_Y g(t)\, d\mu(t).$$

Suppose $q: Y\times Y \to C$ is a continuous kernel that is positive definite, and let H_q denote the Hilbert space associated with q via the reproducing property. The kernel q is said to be universal on Y when H_q is dense in C(Y), and universal on $C_0(Y)$ when Y is non-compact and H_q is dense in $C_0(Y)$.

To every probability measure $\mu \in M(Y)$ we associate, whenever the integral converges in H_q, the element

$$E(\mu) := \int_Y q(\cdot,t)\, d\mu(t) \in H_q.$$

If the mapping $E: M(Y) \to H_q$ is injective, the kernel q is called discriminative. In this situation, E provides a faithful embedding of probability measures into H_q.

For every $h \in H_q$ and $\mu \in M(Y)$, the reproducing identity yields

$$E_\mu(h) = \langle E(\mu), h \rangle_{H_q},$$

so expectations can be computed via inner products in H_q. When q is discriminative, the feature map $\Theta: Y \to H_q$ given by $\Theta(y) := q(\cdot,y)$ is injective and its image is linearly independent.

If Y is compact, every universal kernel is automatically strictly positive definite and discriminative. Moreover, the distance on M(Y) induced by the embedding E metrizes the weak-* topology, so convergence of measures is equivalent to convergence in norm in H_q.

We now recall several consequences of spectral kernel theory in the compact setting.

Let Y be a compact Hausdorff space equipped with a finite Borel measure λ having full support. Assume $q: Y\times Y \to C$ is continuous and positive definite, and let H_q be its associated Hilbert space. Then the following statements hold:[17].

1. The inclusion $H_q \hookrightarrow C(Y)$ is continuous and compact.
2. The operator

$$(Qf)(x) := \int_Y q(x,y) f(y)\, d\lambda(y) \qquad (12)$$

defines a compact linear map from $L^2(Y)$ into H_q with dense image.

3. The adjoint $Q^\dagger: H_q \to L^2(Y)$ coincides with the restriction of the natural embedding $H_q \subset L^2(Y)$.
4. The operator $S := Q^\dagger Q$ is compact, self-adjoint, and positive on $L^2(Y)$, and admits an orthonormal basis $\{\varphi_n\}_{n\ge 0}$ of eigenfunctions with eigenvalues $\{\sigma_n\}_{n\ge 0}$ satisfying $\sigma_n \downarrow 0$.
5. For every index with $\sigma_n > 0$, the functions [18].

$$\psi_n := \sigma_n^{-1/2} Q\varphi_n$$

form an orthonormal basis of H_q.

6. The kernel q admits the representation

$$q(x,y) = \sum_{\sigma_n > 0} \psi_n(x) \overline{\psi_n(y)},$$

where the series converges uniformly on $Y\times Y$.

If all eigenvalues σ_n are strictly positive, then q is universal.

We now restrict attention to kernels invariant under translations on abelian groups.

Let G be a locally compact abelian group with Haar measure ν. Suppose ω is a strictly positive function in $L^1(\hat{G})$ whose inverse Fourier transform κ belongs to $L^1(G)$. Define a kernel $K: G\times G \to C$ by

$$K(x,y) := \kappa(x-y).$$

Then the following properties are satisfied:

1. The kernel K is uniformly continuous, and its associated Hilbert space H_ω is continuously embedded in $C_0(G)$. [19]
2. The operator

$$(Kf)(x) := \int_G K(x,y) f(y)\, d\nu(y) \qquad (13)$$

maps $L^\infty(G)$ into bounded functions and sends both $L^1(G)$ and $L^2(G)$ into uniformly continuous functions.

3. For every character $\gamma\in\hat{G}$,

$$K\gamma=\omega(\gamma)\gamma.$$

If G is compact, then $\hat{G}$ is discrete and the characters diagonalize the operator K. Defining

$$\Omega:=\{\gamma\in\hat{G}:\omega(\gamma)>0\},\qquad \alpha(\gamma):=\sqrt{\omega(\gamma)},$$

an orthonormal basis of H_ω is given by

$$\phi_\gamma:=\alpha(\gamma)\gamma,\qquad \gamma\in\Omega.$$

Accordingly, any $f\in H_\omega$ admits an expansion of the form

$$f=\sum_{\gamma\in\Omega} c(\gamma)\gamma,\qquad \sum_{\gamma\in\Omega}\frac{|c(\gamma)|^2}{\omega(\gamma)}<\infty.$$

This leads to the following characterization. For a function $f\in C(G)$, the following statements are equivalent:

1. $f\in H_\omega$.
2. There exists a unique $v\in L^2(\hat{G})$ such that

$$\hat{f}(\gamma)=\alpha(\gamma)v(\gamma)\quad \text{for all}\ \gamma\in\hat{G},\ \text{and}$$

$$\|f\|_{H_\omega}=\|v\|_{L^2(\hat{G})}.$$

3. Results and Discussion

Hilbert Algebras with Reproducing Kernels Generated by Subconvolutive Functions

In this section, G is taken to be a compact abelian group, except where explicitly indicated otherwise. A principal conclusion of this study demonstrates that imposing appropriate convolution-type constraints on the function that characterizes a harmonic Hilbert space ensures the emergence of a robust and well-defined algebraic framework. Let $\lambda\in L^1(\hat{G})$ satisfy the following conditions: [21].

1. $\lambda(\gamma)>0$ for all $\gamma\in\hat{G}$;
2. there exists a constant $C>0$ such that

$$(\lambda*\lambda)(\gamma)\le C\,\lambda(\gamma),\qquad \gamma\in\hat{G};$$

3. λ is symmetric, i.e., $\lambda(\gamma)=\lambda(-\gamma)$.

Under these assumptions, the harmonic Hilbert space H_λ carries the structure of a Banach *-algebra with identity, where the algebra operations are given by pointwise multiplication and the involution is defined by complex conjugation. In addition, H_λ forms a dense subspace of $C(G)$.

The Banach algebra property asserted above can also be deduced from general results on weighted convolution algebras on dual groups. In particular, convolution estimates on $L^2(\hat{G})$ with weights related to $\lambda^{-1/2}$ imply closure under pointwise multiplication after applying the Fourier transform. The proof given below, however, is adapted specifically to the Hilbert space framework and relies on inner product representations rather than L^p-norm inequalities.

A common approach to constructing functions satisfying the convolution inequality in Theorem 4 is to start from weights whose inverses exhibit additive control. If λ^{-1} satisfies a relaxed subadditivity condition, then λ is known to obey the convolution bound above. Nevertheless, these two notions are logically independent, and examples exist where λ is subconvolutive without the inverse being subadditive [22].

The following auxiliary result connects subconvolutivity of square roots with that of the corresponding squared function.

Assume that G is compact and let $\eta\in L^2(\hat{G})$ be a positive-valued, even function satisfying

$$(\eta*\eta)(\gamma)\le C\,\eta(\gamma),\qquad \gamma\in\hat{G}.$$

Then the function $\lambda:=\eta^2$ belongs to $L^1(\hat{G})$ and is subconvolutive.

Proof. Since G is compact, the dual group $\hat{G}$ is discrete and $L^1(\hat{G})\subset L^2(\hat{G})$. Using the convolution identity and the symmetry of η, we compute

$$(\lambda*\lambda)(\gamma)=(\eta^2*\eta^2)(\gamma)=\langle\eta^2,S_\gamma\eta^2\rangle_{L^2(\hat{G})}=\|\eta S_\gamma\eta\|_{L^2(\hat{G})}^2.$$

By applying the L^1-norm bound and the subconvolutivity of η, we obtain

$(\lambda*\lambda)(\gamma)\leq\|\eta S_\gamma \eta\|_{L^1(\hat{G})}^2=(\eta*\eta(\gamma))^2\leq C^2 \eta^2(\gamma)=C^2 \lambda(\gamma)$,

which proves the claim [23].

Justification of Theorem 4

Let $f,g\in H_\lambda$. By the Fourier characterization of H_λ, there exist functions $u,v\in L^2(\hat{G})$ such that

$$Ff=\sqrt{\lambda}\, u, \qquad Fg=\sqrt{\lambda}\, v.$$

Using the convolution identity for Fourier transforms, we have

$$F(fg)(\gamma)=(Ff*Fg)(\gamma)=(\sqrt{\lambda}\, u*\sqrt{\lambda}\, v)(\gamma).$$

An application of the Cauchy–Schwarz inequality yields

$$|F(fg)(\gamma)|^2\leq(\lambda*\lambda)(\gamma)\,(|u|^2*|v|^2)(\gamma).$$

By the subconvolutivity assumption on λ, it follows that

$$\frac{|F(fg)(\gamma)|^2}{\lambda(\gamma)}\leq C\,(|u|^2*|v|^2)(\gamma).$$

Since $|u|^2,|v|^2\in L^1(\hat{G})$, their convolution belongs to $L^1(\hat{G})$, implying that $fg\in H_\lambda$. Hence, H_λ is closed under pointwise multiplication.

Density of H_λ in $C(G)$ follows from the strict positivity of λ and standard universality results for translation-invariant kernels. The involution defined by complex conjugation preserves the norm due to the symmetry of λ, and the constant function 1_G belongs to H_λ, serving as a multiplicative identity. Symmetry of the resulting Banach *-algebra follows from general spectral considerations for reproducing kernel Hilbert algebras [24].

On the Spectra and States Associated with Reproducing Kernel Hilbert Algebras

In contrast to classical function algebras, a reproducing kernel Hilbert algebra H_λ on a compact abelian group G generally fails to satisfy the C^*-identity

$$\|f^\sharp f\|_{C(G)}=\|f\|_{C(G)}^2,$$

which holds for the C^*-algebra $C(G)$ under pointwise multiplication and complex conjugation. Likewise, H_λ does not usually fulfill the strong compatibility condition characteristic of H^*-algebras, and consequently, its regular representation on $B(H_\lambda)$ is not, in general, a *-representation.

Nevertheless, the reproducing kernel structure provides a canonical family of multiplicative linear functionals. For each $x\in G$, the evaluation map $\varepsilon_x:H_\lambda\to\mathbb{C}$ defined by

$$\varepsilon_x(f):=f(x)=\langle k(x,\cdot),f\rangle_{H_\lambda}$$

is continuous and satisfies

$$\varepsilon_x(fg)=\varepsilon_x(f)\varepsilon_x(g), \qquad \varepsilon_x(f^\sharp)=\overline{\varepsilon_x(f)}, \qquad (14)$$

for all $f,g\in H_\lambda$. Each nonzero evaluation functional therefore defines a character on H_λ, and hence an element of the spectrum $\sigma(H_\lambda)$.

Recall that for a compact Hausdorff space G, the spectrum of the C^*-algebra $C(G)$ consists precisely of evaluation functionals at points of G, and the associated Gelfand transform yields a homeomorphic identification between G and $\sigma(C(G))$. The following result shows that an analogous statement holds for reproducing kernel Hilbert algebras.

Let H_λ be a reproducing kernel Hilbert algebra on a compact abelian group G, where $\lambda\in L^1(\hat{G})$ is strictly positive. Then the following assertions hold: [25].

1. The mapping

$$\beta_\lambda:G\to\sigma(H_\lambda), \qquad \beta_\lambda(x):=\varepsilon_x,$$

is a homeomorphism when $\sigma(H_\lambda)$ is endowed with the weak-* topology.

2. Under the identification of G with $\sigma(C(G))$, the Gelfand transform

$$G_\lambda:H_\lambda\to C(\sigma(H_\lambda))$$

coincides with the inclusion $H_\lambda\hookrightarrow C(G)$. Moreover, the operator norm of G_λ equals $\sqrt{\varphi(0)}$, where $\varphi=F^{-1}\lambda$.

The theorem shows that H_λ and $C(G)$ share the same maximal ideal space, despite the fact that their algebraic norms differ. This result parallels earlier spectral equivalence results for weighted convolution algebras, but is obtained here through direct use of reproducing kernel techniques.

Several consequences follow immediately.

Let $f\in H_\lambda$. Then:

1. If $f(x)\neq 0$ for all $x\in G$, then f is invertible in H_λ.
2. If f is strictly positive, then there exists a strictly positive $g\in H_\lambda$ such that $f=g^2$.
3. The spectrum $\sigma_\lambda(f)$ coincides with the range of f.

Proof. If f vanishes nowhere on G, then $\varepsilon_x(f)\neq 0$ for all x, and hence f does not belong to any maximal ideal of H_λ, implying invertibility. This proves (i).

For (iii), it is clear that $\mathrm{ran}(f)\subset\sigma_\lambda(f)$. Conversely, if $z\notin\mathrm{ran}(f)$, then $f-z$ is nowhere zero and hence invertible, so $z\notin\sigma_\lambda(f)$. Thus, $\sigma_\lambda(f)=\mathrm{ran}(f)$.

Finally, (ii) follows from the functional calculus for unital Banach *-algebras applied to elements with strictly positive spectrum [26].

The algebra H_λ is semisimple and symmetric.

Proof. Semisimplicity follows from injectivity of the Gelfand transform. Symmetry is a direct consequence of the identity $\sigma_\lambda(f^\sharp f)=\mathrm{ran}(f^\sharp f)\subset[0,\infty)$.

We now turn to the description of states. The state space $S(H_\lambda)$ consists of positive linear functionals $\phi:H_\lambda\to\mathbb{C}$ satisfying $\phi(1_G)=1$. Every nonzero evaluation functional ε_x defines a state, whose norm is given by $\|k(x,\cdot)\|_{H_\lambda}$.

Let $P(G)$ denote the set of Borel probability measures on G. Using the kernel mean embedding, we define a map

$$P:P(G)\to S(H_\lambda),\qquad (P\nu)(f):=\int_G f(x)\,d\nu(x). \qquad (15)$$

This map is injective and weak-* continuous.

In addition, each point $x\in G$ induces a state on the noncommutative C^*-algebra $B(H_\lambda)$ via the rank-one projection

$$\Pi_x f:=\frac{f(x)}{k(x,x)}\,k(x,\cdot).$$

This assignment extends to probability measures and yields a map

$$Q:P(G)\to S(B(H_\lambda)).$$

With the above notation, the following statements hold:

1. The maps P and Q are injective and weak-* continuous.
2. For every $\nu\in P(G)$ and $f\in H_\lambda$,

$$\int_G f(x)\,d\nu(x)=P(\nu)(f)=Q(\nu)(\pi(f)),$$

where π denotes the regular representation of H_λ on $B(H_\lambda)$.

The states in the range of Q may be interpreted as classical states of the operator algebra $B(H_\lambda)$ induced by probability measures on G. This operator-valued viewpoint extends standard RKHS constructions and has found applications in areas such as noncommutative probability and quantum information theory [27].

Interactions Between Reproducing Kernel Hilbert Algebras and Markov Semigroups

In this section, we investigate one-parameter families of reproducing kernel Hilbert algebras generated by Markov semigroups acting on $L^2(G)$. Throughout, we assume that G is a compact abelian group and that the Haar measure μ is normalized to a probability measure.

A strongly continuous family $\{T_t\}_{t\geq 0}$ of bounded linear operators on $L^2(G)$ is called a Markov semigroup if, for every $t\geq 0$, the operator T_t preserves positivity, satisfies $T_t 1_G=1_G$, and leaves the integral invariant, i.e.,

$$\int_G T_t f\,d\mu=\int_G f\,d\mu \quad \text{for all } f\in L^2(G).$$

The semigroup is said to be ergodic if the only functions invariant under all T_t are constants.

Let $\{\lambda_t\}_{t>0}\subset L^1(\hat{G})$ be a family of real-valued functions on the dual group satisfying

$$\lambda_t(0)=1,\qquad 0<\lambda_t(\gamma)<1 \ \text{for}\ \gamma\neq 0,\qquad \lambda_{t+s}(\gamma)=\lambda_t(\gamma)\lambda_s(\gamma),\qquad \lambda_t(\gamma)=\lambda_t(-\gamma), \qquad (16)$$

for all $t,s>0$ and $\gamma\in\hat{G}$. Define $\varphi_t:=\mathcal{F}^{-1}\lambda_t$ and the associated translation-invariant kernels

$$k_t(x,y):=\varphi_t(x-y).$$

Let H_t denote the reproducing kernel Hilbert space induced by k_t, and let $K_t:L^2(G)\to L^2(G)$ be the corresponding integral operators.

For each character $\gamma\in\hat{G}$, the mapping $t\mapsto\lambda_t(\gamma)$ is continuous and converges to 1 as $t\to 0^+$. Consequently, the operators K_t converge strongly to the identity on $L^2(G)$. Moreover, $\{K_t\}_{t\ge 0}$ forms a strongly continuous semigroup of self-adjoint contractions consisting of compact operators.

By standard semigroup theory, there exists a positive self-adjoint operator D on $L^2(G)$ such that

$$K_t=e^{-tD},\qquad t\ge 0.$$

The operator D is diagonal with respect to the character basis, i.e.,

$$D\gamma=\omega(\gamma)\gamma,\qquad \omega(\gamma)=-\frac{1}{t}\log\lambda_t(\gamma),$$

and satisfies $\omega(0)=0$. It follows that $-D$ is the generator of an ergodic Markov semigroup. For $t>0$, the kernel $k_t(x,\cdot)$ represents a transition probability density:

$$k_t(x,y)\ge 0,\qquad \int_G k_t(x,y)\,d\mu(y)=1.$$

We now characterize when the spaces H_t inherit a Banach algebra structure.

Assume that the family $\{\lambda_t\}_{t>0}$ satisfies (16). Then the corresponding RKHSs H_t are reproducing kernel Hilbert algebras if and only if, for every $t>0$, the function λ_t is subconvolutive, i.e., there exists $C_t>0$ such that

$$(\lambda_t*\lambda_t)(\gamma)\le C_t\,\lambda_t(\gamma),\qquad \gamma\in\hat{G}.$$

On the d-dimensional torus $G=T^d$ with dual group $\hat{G}\cong Z^d$, the family

$$\lambda_t(\gamma)=e^{-t|\gamma|^\alpha},\qquad \alpha\in(0,2],$$

satisfies the conditions above. In this case, the generator D coincides with a fractional power of the Laplacian, and $\{H_t\}_{t>0}$ forms a family of reproducing kernel Hilbert algebras [28].

Proof of Theorem 11

The sufficiency follows directly from the general algebra criterion established in Section 4. We therefore focus on the necessity. Assume that, for a fixed $t>0$, the space H_t is a Banach algebra under pointwise multiplication.

Set $\eta_t:=\lambda_t^{1/2}$. We first record the following auxiliary fact.

If H_t is closed under pointwise multiplication, then for any $u,v\in L^2(\hat{G})$ there exists a unique $w\in L^2(\hat{G})$ such that

$$\eta_t w=(\eta_t u)*(\eta_t v).$$

Proof. By the Fourier characterization of H_t, there exist functions $f,g\in H_t$ with $Ff=\eta_t u$ and $Fg=\eta_t v$. Since H_t is a Banach algebra, $fg\in H_t$, and hence there exists $w\in L^2(\hat{G})$ such that $F(fg)=\eta_t w$. The claim follows from the identity $F(fg)=(Ff)*(Fg)$.

Since G is compact, $\eta_t\in L^1(\hat{G})\cap L^2(\hat{G})$. Applying the lemma with $u=v=\eta_t$, we obtain

$$(\eta_t*\eta_t)(\gamma)=\eta_t(\gamma)\,w(\gamma),$$

for some bounded function w. Taking limits and using monotone convergence yields

$$(\eta_t*\eta_t)(\gamma)\le C_t\,\eta_t(\gamma),$$

which implies

$$(\lambda_t*\lambda_t)(\gamma)=(\eta_t^2*\eta_t^2)(\gamma)\le C_t^2\,\lambda_t(\gamma).$$

Thus, λ_t is subconvolutive. Since $t>0$ was arbitrary, the result follows.

Algebraic Frameworks Arising from Fourier–Wermer Theory

Motivated by problems arising in high-dimensional approximation theory, we conclude the paper by examining the relationship between reproducing kernel Hilbert algebras and a class of weighted Fourier algebras on compact abelian groups, commonly referred to as Fourier–Wermer algebras [22].

Let $w:\hat{G}\to(0,\infty)$ be a positive weight function. We define the space

$$A_w:=\{f\in L^1(G):\ \sum_{\gamma\in\hat{G}} w(\gamma)\,|Ff(\gamma)|<\infty\},$$

equipped with the norm

$$\|f\|_{A_w}:=\sum_{\gamma\in\hat{G}} w(\gamma)\,|Ff(\gamma)|.$$

If $G=\mathbb{T}^d$, spaces of this form arise naturally in numerical schemes for approximating multivariate periodic functions and are frequently used in settings where the dimension d is large.

Assuming that w is bounded away from zero, A_w embeds continuously into the classical Wiener algebra

$$A(G):=\{f\in L^1(G): \sum_{\gamma\in\hat{G}} |Ff(\gamma)|<\infty\},$$

which consists of continuous functions with absolutely summable Fourier coefficients. Equivalently, A_w can be viewed as the inverse Fourier image of the weighted sequence space $L_w^1(\hat{G})$, that is, [24].

$$A_w=F^{-1}(L_w^1(\hat{G})).$$

More generally, for $1<p<\infty$, we denote by $L_w^p(\hat{G})$ the space of functions $\hat{f}$ on $\hat{G}$ with norm

$$\|\hat{f}\|_{L_w^p(\hat{G})}:=\left(\sum_{\gamma\in\hat{G}} |w(\gamma)\hat{f}(\gamma)|^p\right)^{1/p}.$$

A central problem in approximation theory concerns the efficient approximation of functions belonging to an input space such as $A_w(G)$ by elements of finite-dimensional subspaces, with error measured in a target space into which $A_w(G)$ embeds continuously. In high-dimensional regimes, the key question is how approximation errors scale with both the dimension and the size of the approximating subspace, and whether algebraic or smoothness structure can be exploited to mitigate dimensional effects [26].

Recent results show that, for certain weights encoding mixed smoothness properties, the optimal approximation error decays at an almost polynomial rate in the dimension. However, the availability of a Banach algebra structure for $A_w(G)$ plays an essential role in designing approximation schemes that preserve nonlinear operations such as multiplication.

Algebra Structure of A_w

We now address the question of when A_w forms a Banach algebra under pointwise multiplication. A sufficient condition is given by a convolution bound on the inverse weight. Specifically, if there exists a constant $C>0$ such that

$$(w^{-1}*w^{-1})(\gamma)\le C\, w^{-1}(\gamma), \qquad \gamma\in\hat{G}, \qquad (17)$$

then $L_w^1(\hat{G})$ is a convolution algebra, and hence A_w is a Banach algebra under pointwise multiplication.

Under this assumption, A_w is a dense subalgebra of $A(G)$. Moreover, spectral invariance results for weighted convolution algebras imply that the maximal ideal space of A_w contains a homeomorphic copy of the group G. In fact, when the weight w satisfies suitable regularity assumptions, the spectrum of A_w coincides with that of $C(G)$.

Weights arising from dominating mixed smoothness provide an important example. For such weights, the condition (17) holds whenever the smoothness parameter exceeds a critical threshold, ensuring that A_w is a Banach algebra on $\mathbb{T}^d$.

More generally, if w is subadditive, then weighted sequence spaces of the form

$$\mathcal{L}_w^p(\hat{G}):=L^1(\hat{G})\cap L_w^p(\hat{G}), \qquad 1\le p\le\infty,$$

equipped with the norm

$$\|\hat{f}\|_{\mathcal{L}_w^p}:=\|\hat{f}\|_{L^1}+\|\hat{f}\|_{L_w^p},$$

are Banach convolution algebras. As a consequence, their Fourier images inherit Banach algebra structure, and spectral equivalence with $C(G)$ follows from standard results in commutative Banach algebra theory [27].

Embedding Relations with RKHAs

We now consider the situation in which both A_w and a reproducing kernel Hilbert algebra H_λ coexist, with $\lambda=w^{-2}$ satisfying the assumptions ensuring that H_λ is a Banach *-algebra.

In this case, A_w embeds continuously into H_λ. Indeed, writing

$$(Ff)(\gamma)=w^{-1}(\gamma)\,(w(\gamma)Ff(\gamma)),$$

and comparing the defining norms, one obtains

$$\|f\|_{H_\lambda}\le\|f\|_{A_w},$$

so the inclusion map $A_w \hookrightarrow H_\lambda$ is continuous with norm one. Consequently, multiplication by elements of A_w defines bounded operators on H_λ, yielding a faithful representation of A_w into $B(H_\lambda)$.

In the opposite direction, a continuous embedding of H_λ into A_w cannot be expected in general, since $L^2(\hat{G})$ does not embed into $L^1(\hat{G})$ unless $\hat{G}$ is finite. However, by relaxing the regularity slightly, one can obtain continuous embeddings of the form

$$H_{\lambda^{(1+\varepsilon)}} \hookrightarrow A_w, \qquad \varepsilon>0,$$

for suitable families of weights, such as those arising from Markov semigroups. These embeddings establish a precise link between Hilbertian and l^1-type Fourier regularity, and are particularly useful in applications where both algebraic closure and approximation efficiency are required [28-30].

4. Conclusion

This work develops a unified harmonic-analytic framework for reproducing kernel Hilbert algebras on compact abelian groups. By exploiting translation-invariant positive definite kernels and Fourier duality, we characterized the structure of the associated Hilbert spaces and identified precise conditions ensuring stability under pointwise multiplication and involution. Subconvolutive conditions on spectral weights were shown to play a central role in guaranteeing Banach $*^{*}*$-algebra properties and spectral symmetry. The results further establish a canonical identification between the spectrum of reproducing kernel Hilbert algebras and the underlying group, paralleling classical results for C(G)C(G)C(G). Applications to Markov semigroups and weighted Fourier algebras illustrate how these structures naturally arise in harmonic analysis and approximation theory. Overall, the study highlights reproducing kernel Hilbert algebras as a robust bridge between kernel methods, Fourier analysis, and algebraic structures

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