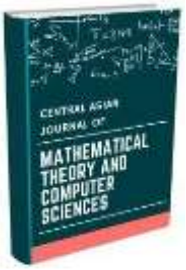

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Article

A Coarse Embedding of n-Point Persistence Diagram Spaces into Hilbert Spaces

Husham Sabhan Khalaf

1. College of Education for Pure Sciences, University of Maragheh, Maragheh, Iran

* Correspondence: almhymydhsham@gmail.com

Abstract: We establish that the metric space of persistence diagrams consisting of exactly n points admits a coarse embedding into a Hilbert space when equipped with either the bottleneck distance or a Wasserstein distance. This result is obtained by demonstrating that such spaces possess asymptotic dimension equal to 2n. As a consequence, a broad range of analytical and geometric tools native to Hilbert spaces become applicable to the study of persistence diagrams with bounded cardinality. In contrast, we show that if the number of points in persistence diagrams is allowed to grow without bound, the associated metric spaces fail to have finite asymptotic dimension. Moreover, for the bottleneck metric, we prove that the corresponding unbounded persistence diagram space does not admit a coarse embedding into any Hilbert space.

Keywords: Persistence diagrams; Coarse embedding; Hilbert spaces; Asymptotic dimension

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1. Introduction

Persistent homology is a homological framework designed to capture topological features of data across multiple scales. One of its principal outputs is the persistence diagram, which encodes the birth and death of homological features in a concise planar representation. This representation enjoys two fundamental advantages: first, its inherently two-dimensional structure facilitates visualization and qualitative analysis; second, it exhibits stability with respect to perturbations of the input data when equipped with suitable metrics, most notably the bottleneck and Wasserstein distances (see [1]). These characteristics have been fundamental in driving the swift advancement of persistent homology across both theoretical research and practical data-analytic applications.

Despite these advantages, a significant challenge arises from the fact that most statistical and machine learning methodologies are formulated within the framework of Hilbert spaces. Any embedding of this nature would form a conceptual and practical bridge between topological data analysis and the rich toolbox of statistical learning theory.

Previous work on embeddings of persistence diagram spaces has largely produced negative results. It has been shown that, for various classes of persistence diagram spaces, isometric embeddings [2], bilipschitz embeddings [3], and even coarse embeddings [4] into Hilbert spaces do not exist (see also Remark 4.4). More generally, foundational geometric and functional-analytic properties of spaces of persistence diagrams, including explanations for why these spaces fail to be Hilbert spaces themselves, have been established in several works such as [5, 6].

In the present paper, we focus on coarse embeddings, that is, embeddings that preserve large-scale geometry while allowing controlled distortions at small scales. In light of the aforementioned negative results, coarse embeddings emerge as the most promising setting in which positive results might still be achievable. The conceptual framework for coarse geometry—also known as asymptotic topology—originated in geometric group theory, particularly in the work of Gromov [7]. Interest in this area intensified following Yu's seminal result [8],

This work established that having finite asymptotic dimension—or, more broadly, admitting a coarse embedding into a Hilbert space—leads to profound implications for questions such as the Novikov conjecture, thereby initiating widespread investigation into coarse invariants and the large-scale geometry of metric spaces.

The principal contributions of this paper can be summarized as follows.

Theorem 3.2. For persistence diagram spaces containing precisely nnn points and endowed with either the bottleneck metric or a Wasserstein metric, the asymptotic dimension is equal to 2n. As a direct implication, these spaces admit a coarse embedding into a Hilbert space.

This result constitutes, to our knowledge, the first affirmative embedding theorem for spaces of persistence diagrams.

Theorem 4.3. The collection of persistence diagrams with finite cardinality, when equipped with the bottleneck distance, fails to admit a coarse embedding into any Hilbert space.

An additional technical contribution of this work is a reformulation of the metrics on spaces of persistence diagrams presented in Section 2. Although this formulation departs from standard Euclidean intuition, it yields more concise metric descriptions and enables an efficient application of coarse-geometric techniques. This approach often leads to significantly streamlined arguments, as illustrated in Section 3 (see Remarks 2.5 and 2.7). Furthermore, when the cardinality of persistence diagrams is unbounded, the resulting space fails to have finite asymptotic dimension, as shown in Corollary 2.13.

Related work.

A variety of mappings—often referred to as kernels—from persistence diagram spaces into Hilbert spaces are currently employed in applications. Several such constructions are surveyed in [9], where the possibility of bilipschitz embeddings is also analyzed. It was shown in [10] that spaces of persistence diagrams with finitely many points do not satisfy property A when equipped with Wasserstein metrics, implying the absence of finite asymptotic dimension. Results closely related to Theorem 4.3 appear in [11], which establishes the non-existence of coarse embeddings for persistence diagrams with countably many points under the bottleneck metric. Moreover, Wagner [12] demonstrated that analogous non-embedding results hold for Wasserstein metrics with parameter $p>2$.

The computation of asymptotic dimension in this work is based on a theorem of Kasprowski [13] (Theorem 2.15), which shows that finite group actions preserve asymptotic dimension. The actions considered in our setting arise as a special case of coarsely n-to-1 maps, for which effective bounds on asymptotic dimension are known [14]. Furthermore, we expect that the techniques developed here can be adapted to simplify asymptotic dimension computations in hyperspaces, potentially yielding alternative proofs of the results in [15].

Structure of the paper.

Section 2 provides the essential preliminaries on persistence diagrams, introducing an alternative yet practically useful formulation, along with the fundamental ideas of large-scale geometry. Section 3 then focuses on determining the asymptotic dimension associated with persistence diagram spaces consisting of exactly nnn points. Section 4 examines embeddings of finite metric spaces into persistence diagram spaces equipped with the bottleneck distance and culminates in demonstrating that these spaces fail to admit coarse embeddings into Hilbert space.

Note on this version.

In an earlier version of this work, I claimed that a direct application of Kasprowski's theorem [16] yields the asymptotic dimension of the persistence diagram space consisting of nnn points.However, the space under consideration was not proper, a requirement for the direct use of that theorem. The proof has since been corrected, and the statement of Theorem 3.2 remains valid in the present version.

2. Materials and Methods

Preliminary Definitions and Concepts

This section introduces the notation and foundational concepts required throughout the paper. We begin by formalizing the notion of persistence diagrams and the metrics imposed on their associated spaces. We then review essential notions from coarse geometry, which play a central role in our analysis.

Persistence Diagram Framework

Persistence diagrams serve as two-dimensional representations that encode the essential features of persistence modules and persistent homology. We begin by developing a unified framework that encompasses the principal classes of persistence diagram spaces relevant to the present study [17]. To this end, we introduce the following objects:

1. The metric d_∞ on R^2, defined by
   $d_\infty((x_1,x_2),(y_1,y_2))=\max\{|x_1-y_1|,|x_2-y_2|\}$.
2. The space $D_1=T\cup\{\Delta\}$, where
   $T=\{(x_1,x_2)\in R^2 \mid x_2>x_1\geq 0\}$,

and $\Delta\notin T$.

3. A semi-metric δ on D_1, extending d_∞ from T by setting
   $\delta((x_1,x_2),\Delta)=(x_2-x_1)/2$.

The subset T may equivalently be replaced by R^2 or by the region

$\{(x_1,x_2)\in R^2 \mid x_2>x_1\}$.

We adopt the present choice as it is consistent with standard usage in the literature; moreover, all arguments and conclusions developed in this work remain unchanged under these alternative formulations.

The distinguished element Δ corresponds to the diagonal

$\{(x_1,x_2)\in R^2 \mid x_1=x_2\}$,

which plays a central role in the classical representation of persistence diagrams. The value $\delta((x_1,x_2),\Delta)$ agrees with the d_∞-distance from the point (x_1,x_2) to the diagonal. Treating the entire diagonal as a single point provides a technically convenient simplification while remaining fully equivalent to the conventional formulation involving infinitely many diagonal points.

Fix $n\in N$. We introduce the following notions.

A matching is defined as a bijective correspondence between two finite sets. In the special case where the sets coincide, a matching reduces to a permutation.

For a fixed integer n, we define the collection of persistence diagrams containing no more than n points as the quotient space

$D_n (D_1^{(n)})/S_n$,

where the symmetric group S_n acts on the Cartesian product $D_1^{(n)}$ by reordering its components. Two elements

$z=(z_1,\ldots,z_n)$ and $z'=(z'_1,\ldots,z'_n)$

are regarded as equivalent whenever there exists a permutation $\varphi\in S_n$ satisfying

$z_i=z'_{\varphi(i)}$ foreachi=1,…,n.

$z_i=z'_{\varphi(i)}$ foralli.

There is a canonical embedding $D_n\hookrightarrow D_{n+1}$ obtained by adjoining the diagonal element Δ. In what follows, this inclusion is used implicitly.

$D_{<\infty}=\bigcup_{n\in N} D_n$.

Let $n\in N$ and $p>1$.

The bottleneck distance d_B on D_n is defined as follows. For $z,z'\in D_n$, append n copies of Δ to obtain elements of $(D_1)^{2n}$ and set

$$d_B(z,z')=\min_{\varphi\in S_{2n}} \max_i \delta(z_i, z'_{\varphi(i)}).$$

Denote (D_n,d_B) by D_n^B and $(D_{<\infty},d_B)$ by $D_{<\infty}^B$.

The p-Wasserstein distance on D_n is given by

$$d_{W,p}(z,z')=\min_{\varphi\in S_{2n}} \left(\sum_i \delta(z_i, z'_{\varphi(i)})^p\right)^{1/p}.$$

We write $D_n^{W,p}=(D_n,d_{W,p})$ and $D_{<\infty}^{W,p}=(D_{<\infty},d_{W,p})$.

The conventions above ensure that the distances are well-defined even when comparing diagrams with different numbers of points.

The definitions above reformulate the standard partial-matching approach [18] using finite permutations. This avoids the explicit use of infinitely many diagonal points and leads to more concise expressions, which are particularly advantageous for coarse-geometric arguments.

For later use, we introduce an equivalent formulation of the bottleneck metric.

1. For $z,z'\in D_1$, define

$$d_B^1(z,z')=\min\{d_\infty(z,z'),\max\{\delta(z,\Delta),\delta(z',\Delta)\}\}.$$

2. For $z,z'\in D_n$, define

$$\tilde{d}_B(z,z')=\min_{\varphi\in S_n} \max_i d_B^1(z_i, z'_{\varphi(i)}).$$

The metrics d_B and $\tilde{d}_B$ coincide. Moreover, this formulation allows D_n^B to be viewed as a quotient of $(D_1,d_B^1)^n$ by the finite group S_n, a fact crucial for computing asymptotic dimension in Section 3.

Large-Scale Geometric Framework

We now outline the principal notions from coarse geometry that will be used repeatedly in this paper.

Consider a function $f:X\to Y$ between two metric spaces.

1. The map f is called a coarse embedding if there exist monotone functions $\varrho_1,\varrho_2:[0,\infty)\to[0,\infty)$ such that, for all $x,x'\in X$,

$$\varrho_1(d(x,x')) \le d(f(x),f(x')) \le \varrho_2(d(x,x')),$$

and moreover the lower control function satisfies

$$\lim_{t\to\infty} \varrho_1(t)=\infty.$$

2. If, moreover, the image $f(X)$ is coarsely dense in Y, then f is called a coarse equivalence.

A metric space X is said to have asymptotic dimension at most n, denoted $\operatorname{asdim} X\le n$, if for every $R>0$ the space X admits a decomposition into $n+1$ families of subsets that are uniformly bounded and pairwise R-disjoint. Asymptotic dimension is preserved under coarse equivalence [19]. Typical examples include $\operatorname{asdim}(\mathbb{R})\le 1$ and $\operatorname{asdim}(\mathbb{R}^m)=m$.

Let X and Y be subspaces of a metric space.

1. Union Theorem.

$$\operatorname{asdim}(X\cup Y)=\max\{\operatorname{asdim}X,\operatorname{asdim}Y\}.$$

2. Product Theorem.

$$\operatorname{asdim}(X\times Y)\le\operatorname{asdim}X+\operatorname{asdim}Y$$

To derive lower bounds for asymptotic dimension, we employ the following criterion. Let $p>1$. If for every $R>0$ the cube $([0,R]^m,d_\infty)$ or $([0,R]^m,d_p)$ admits an isometric embedding into a metric space X, then $\operatorname{asdim} X\ge m$.

As a consequence, for any $p>1$, the spaces $D_{<\infty}^B$ and $D_{<\infty}^{W,p}$ fail to have finite asymptotic dimension. On the other hand, any metric space of finite asymptotic dimension admits a coarse embedding into Hilbert space [20].

We will also rely on several results concerning finite group actions and coarse non-embeddability. When a finite group operates on a proper metric space through distance-preserving transformations, the induced quotient space retains exactly the same asymptotic dimension as the original space [14].

Moreover, the coarse disjoint union of the spaces $(Z_n)^m$, equipped with the max metric, does not admit a coarse embedding into Hilbert space [15]. Finally, a metric space admits a coarse embedding into Hilbert space if and only if this property holds uniformly over all of its finite subsets [16].

3. Results and Discussion

Results of Obstructions to Coarse Embedding

In this part of the paper, we analyze how finite metric spaces can be mapped into bottleneck persistence diagram spaces D_n^B and show that the space $D_{(<\infty)}^B$ does not admit a coarse embedding into Hilbert space. For related results on embeddings of separable bounded metric spaces, we refer to [16, Theorem~3.1].

For every $h\in N$, any finite metric space (X,d) admits an isometric embedding into $D_{(|X|)}^B$ such that the image is contained entirely above the horizontal line of height h.

Proof. Let $X=\{x_0,x_1,\ldots,x_n\}$ be a finite metric space and denote its diameter by $R=diam(X)$. We construct a map $f:X\to D_{(|X|)}^B$ as follows. For each element $x_k\in X$, set

$$f(x_k)=\{(3Ri,3Ri+3R+d(x_k,x_i)+h) \mid i=1,2,\ldots,n\}.$$

We show that this map preserves distances.

Observe first that, for every fixed k, the image $f(x_k)$ contains precisely one point whose first coordinate equals 3Ri for each index i. Next, for any two indices j and k, the optimal matching between the diagrams $f(x_j)$ and $f(x_k)$ is necessarily perfect and pairs points having the same first coordinate. For arbitrary indices i,j,k, we compute

$$d_\infty((3Ri,3Ri+3R+d(x_j,x_i)+h),(3Ri,3Ri+3R+d(x_k,x_i)+h))=|d(x_j,x_i)-d(x_k,x_i)|\le d(x_j,x_k),$$

with equality attained whenever i=j or i=k. Consequently, for all j,k we obtain

$$d_B(f(x_j),f(x_k))=d(x_j,x_k).$$

Hence the mapping f preserves pairwise distances and is therefore an isometric embedding.

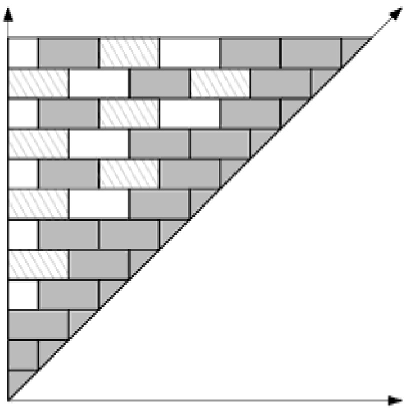

Figure 1. Asdim $D_B^1 \le 2$

Let $\{A_i\}_{(i\in N)}$ be a collection of finite metric spaces. Their coarse disjoint union admits an isometric embedding into $D_{(<\infty)}^B$. Moreover, if there exists $M\in N$ such that $|A_i|\le M$ for all i, then the image is contained in D_M^B.

Proof. By Lemma 4.1, each A_i can be embedded isometrically into $D_{(|A_i|)}^B$ above an arbitrarily chosen height. We construct the embeddings inductively so that the vertical coordinates of the image of A_i lie at least $\max_{(j\le i)} diam(A_j)$ above those of $A_{(i-1)}$. Denoting by $\tilde{A}$ the union of the embedded images, the induced subspace metric turns $\tilde{A}$ into a coarse disjoint union of the spaces $\{A_i\}$. If $|A_i|\le M$ for all i, Lemma 4.1 ensures that each A_i embeds into D_M^B, hence $\tilde{A}\subset D_M^B$.

The bottleneck persistence diagram space $D_{(<\infty)}^B$ cannot be coarsely embedded into any Hilbert space.

Proof. This conclusion follows by combining the obstruction provided by Theorem 2.16 with the embedding construction given in Corollary 4.2. Indeed, Corollary 4.2 ensures that the coarse disjoint union of the metric spaces $((Z_m)^n,d_\infty)$ can be realized isometrically inside $D_{(<\infty)}^B$. Since Theorem 2.16 asserts that such a coarse disjoint union does not admit a coarse embedding into Hilbert space, the same must hold for $D_{(<\infty)}^B$. As a consequence, $D_{(<\infty)}^B$ cannot have finite asymptotic dimension.

While an earlier version of this article was under preparation, an independent manuscript [12] appeared that established closely related negative results concerning coarse embeddability. The ambient space considered in [14] contains $D_{(<\infty)}^B$ as a subspace, and thus its non-embeddability outcome is already implied by the theorem above. Related ideas and constructions have also been discussed in the study of hyperspaces, see for instance [15].

The finite-determination criterion for coarse embeddability into Hilbert space (Theorem 2.17) provides an alternative route to proving the non-embeddability of $D_{(<\infty)}^B$. Indeed, if $D_{(<\infty)}^B$ admitted a coarse embedding into Hilbert space, then by Lemma 4.1 any finite subset of a metric space Y that does not coarsely embed into l_2 (for example l_p with $p>2$, see [9]) could be viewed as a finite subset of $D_{(<\infty)}^B$, leading to a contradiction. Using this approach, Wagner [11] showed that $D_{(<\infty)}^{(W,p)}$ does not coarsely embed into Hilbert space for $p>2$.

Let (X,d) be a metric space that admits a coarse embedding into Hilbert space, and let (Y,D) be a metric space whose finite subsets uniformly coarsely embed into X. Then Y itself admits a coarse embedding into Hilbert space.

If a metric space (X,d) contains isometric copies of all finite metric spaces, then (X,d) does not coarsely embed into Hilbert space. In particular, $D_{(<\infty)}^B$ fails to coarsely embed into Hilbert space.

Finite subsets of l_p uniformly coarsely embed into $D_{(<\infty)}^{(W,p)}$. Consequently, $D_{(<\infty)}^{(W,p)}$ does not coarsely embed into Hilbert space for $p>2$ [3].

4. Conclusion

This study shows that every finite metric space admits an explicit isometric embedding into a bottleneck persistence diagram space DǀXǀBD_{∣X∣}^BDǀXǀB, with full preservation of pairwise distances. Using this construction, coarse disjoint unions of finite metric spaces can be realized isometrically inside D(<∞)BD_{(<\infty)}^BD(<∞)B. As a consequence, D(<∞)BD_{(<\infty)}^BD(<∞)B contains isometric copies of all finite metric spaces. This universality yields a strong obstruction to coarse embeddability into Hilbert space. Therefore, D(<∞)BD_{(<\infty)}^BD(<∞)B does not admit a coarse embedding into any Hilbert space and, in particular, cannot have finite asymptotic dimension. These results clarify intrinsic large-scale geometric limitations of bottleneck persistence diagram spaces.

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