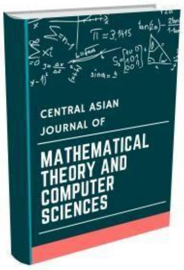

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Article

Flexible Bayesian Estimation of the Weibull Survival Function Under Censored Data

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Abstract: This paper aims at estimating the Weibull survival function under the right-censored data using five Bayesian estimation procedures. The Weibull distribution is extensively used in survival analysis because it can be flexible in its ability to model the various variation of hazard rate. Since the closed form Bayesian estimates of the survival tend to be unavailable, the techniques of simulation are needed. A whole Monte Carlo simulation experiment is carried out to provide an estimate of the Weibull survival function and evaluate the performance of estimators. It is compared in terms of accuracy, the mean squared error and also in terms of credible interval coverage. The impact of sample size and proportion of censoring is also examined. The paper determines the most effective Bayesian estimator to be used in estimating the Weibull survival functions.

Keywords: Bayesian Survival Analysis, Weibull Distribution, Survival Function Estimation, Right-Censored Data, Monte Carlo Simulation, Bayesian Estimation Methods

1. Introduction

Survival analysis is a basic field of statistical modelling that deals with the analysis of time-to-event data, the event of interest allows to be death, failure, relapse or any other event that terminates [1]. One of the most important numbers in survival analysis would be the survival function, the probability of an individual or system surviving up to a certain time. The survival function is important in numerous practical applications, such as medical research, engineering reliability, economics, and financial risk analysis where it has been shown that accurate estimation of the survival function is vital to the applications of the model [2]. Weibull distribution is prominent among the parametric survival models because of its analytical tractability, and flexibility. The shape parameter of the Weibull survival function allows it to fit the decreasing, constant or increasing hazard rates, and thus, it is applicable in a large variety of survival processes in real-world scenarios. Consequently, the Weibull model is now a reference model in applied survival analysis when a parametric formulation is suitable, i.e. where the model is parametrical [3].

Practical use Survival data can be incomplete because they can be censored, especially by right censoring where the event of interest is not experienced by all members of the study population. Censoring makes the process of statistical inference more difficult and makes the estimation less precise, particularly in small samples. The classical estimation techniques, e.g. maximum likelihood estimation, can work poorly in the presence of high censoring or when there is scanty data, which drives the adoption of alternative estimation framework in the form of the Bayesian inference [4].

The Bayesian method is the one that considers the parameters of the Weibull distribution as random variables and uses the Bayes theorem to add the already known information with the observed data. This model offers a consistent method of measuring uncertainty and enables one to make direct probabilistic inferences on the uncertainty itself. Nonetheless, the Bayesian estimation of Weibull survival function is analytically difficult, with the posterior distribution of the model parameters mostly being in no closed-form form, especially when there are censored observations [3], [4].

The computational Bayesian statistics have resulted in the creation of a number of numerical methods of estimating posterior distributions. The Markov Chain Monte Carlo techniques continue to be the main foundation of the computation of the Bayesian, which have asymptotically accurate inferences but with a higher computational cost. Hamiltonian Monte Carlo algorithms are more efficient since they utilize gradient information whereas Variational Bayesian algorithms offer more efficient but approximate answers. Bayesian nonparametric approaches like Dirichlet process mixtures, loosen parametric assumptions and enable them to be more flexible in modeling heterogeneous patterns of survival. Also, Approximate Bayesian Computation provides the use of likelihood-free methods where likelihood computation is complicated and unfeasible [4], [5], [6], [7], [8].

Although there is an overwhelming body of literature on Bayesian survival analysis, several studies have nevertheless not examined the systematic comparison of various Bayesian estimation methods with respect to survival function estimation in the Weibull model. Specifically, the simulation-based evidence comparing the performance of various Bayesian approaches is missing in relation to the assessment of their performance at different sample sizes and levels of censoring. These comparisons are necessary to advise applied researchers on the choice of suitable Bayesian tools to use in survival analysis[3], [4].

The purpose of this paper is to fill this gap by performing a Monte Carlo simulation study to compare five Bayesian approximations to the Weibull survival function. It is centered on the position of measuring the accuracy of point and interval estimation of the survival function under a variety of experimental conditions. The final aim is to find the best Bayesian estimator to the estimation of the Weibull survival functions and to give the useful suggestions to the applied survival analysis [6], [9], [10].

2. Materials and Methods

2.1 Bayesian Estimation Using Metropolis–Hastings MCMC

Applying the Bayesian model, the joint posterior of Weibull parameters (λ, k) is computed through the Bayes theorem [11].

$$p(\lambda, k \mid t, \delta) \propto \prod_{i=1}^{n} f\,(t_i \mid \lambda, k)^{\delta_i} S(t_i \mid \lambda, k)^{1-\delta_i}\, \pi(\lambda, k) \qquad (1)$$

$f(t \mid \lambda, k)$,$S(t \mid \lambda, k)$: where f represents the Weibull distribution, and S is the survival curve $S(t \mid \lambda, k) = \exp[-(t/\lambda)^k]$. As the posterior is not closed, Metropolis-Hastening forms a Markov chain by proposing (λ^*, k^*) according to a proposal density $q(\cdot)$ and accepting it with probability [10], [12].

$$\alpha = \min\left\{1, \frac{p(\lambda^*, k^* \mid \text{data})}{p(\lambda^{(s)}, k^{(s)} \mid \text{data})}\right\} \qquad (2)$$

Bayesian survival function estimator is calculated as the posterior mean.

$$\hat{S}(t) = \frac{1}{S} \sum_{s=1}^{S} \exp\left[-\left(\frac{t}{\lambda^{(s)}}\right)^{k^{(s)}}\right] \qquad (3)$$

2.2 Bayesian Estimation Using Hamiltonian Monte Carlo (HMC)

Hamiltonian Monte Carlo is a better explorer of the posterior by reparameterizing $\theta = (\log\lambda, \log k)$ and augmenting the posterior with auxiliary momentum variables p ,resulting in the Hamiltonian function

$$H(\theta, p) = -\log p(\theta \mid \text{data}) + \frac{1}{2} p^{\top} p \qquad (4)$$

Posterior samples are generated by simulating Hamiltonian dynamics governed by

$$\frac{d\theta}{dt} = \frac{\partial H}{\partial p}, \qquad (5)$$

And then Metropolis correction. The resultant posterior draws give estimates of the survival function using.

$$\hat{S}(t) = E\left[\exp\left(-\left(\frac{t}{\lambda}\right)^{k}\right) \mid \text{data}\right] \qquad (6)$$

It is necessary that the convergence is efficient even in the case of correlated parameters [13], [14].

2.3 Bayesian Estimation Using Variational Bayes (VB)

Variational Bayes estimates the intractable posterior $p(\theta \mid \text{data})$ by a tractable distribution $q(\theta)$ by minimizing the Kullback-Leibler divergence [15], [16].

$$q^{*}(\theta) = \underset{q}{\operatorname{argmin}} KL\left(q(\theta) \parallel p(\theta \mid \text{data})\right) \qquad (7)$$

This maximization can be equated to maximization of Evidence Lower Bound (ELBO).

$$ELBO = E_q[\log L(\theta)] + E_q[\log \pi(\theta)] - E_q[\log q(\theta)] \qquad (8)$$

In which $L(\theta)$ is the Weibull censor likelihood. After finding $q^{*}(\theta)$, the survival is estimated through the use of posterior expectation.

$$\hat{S}(t) = E_q\left[\exp\left(-\left(\frac{t}{\lambda}\right)^{k}\right)\right] \qquad (9)$$

To give an approximate and quick Bayesian solution [17], [18].

2.4 Bayesian Nonparametric Estimation Using Dirichlet Process Mixtures

The survival distribution in the Bayesian nonparametric method takes the form of an infinite mixture of Weibull components [19].

$$S(t) = \int \exp\left[-\left(\frac{t}{\lambda}\right)^{k}\right] dG(\lambda, k), G \sim DP(\alpha, G_0) \qquad (10)$$

In which G_0 is a base distribution and 0alpha determines clustering. The weighted mixture is the posterior distribution of G calculated with the help of stick-breaking or Gibbs sampling representations.

$$\hat{S}(t) = \sum_{j=1}^{\infty} w_j \exp\left[-\left(\frac{t}{\lambda_j}\right)^{k_j}\right] \qquad (11)$$

This approach is a less restrictive version of single-Weibull by unwinding the assumption of observed heterogeneity in the survival behavior.

2.5 Bayesian Estimation Using Approximate Bayesian Computation (ABC)

Approximate Bayesian Computation instead of performing a likelihood evaluation involves performing simulation, in which the parameters $\theta = (\lambda, k)$ are sampled with prior $\pi(\theta)$, and synthetic data is obtained using the Weibull model. The observed and the simulated summary statistics are compared to determine that the distance between them satisfies a draw.

$$\rho\left(S^{\text{sim}}(t), S^{\text{obs}}(t)\right) < \varepsilon \qquad (12)$$

For a tolerance ε.The accepted samples are approximations to the posterior distribution[20].

$$p_\varepsilon(\theta \mid \text{data}) \approx p(\theta \mid \text{data}) \quad (13)$$

And the estimator of survival is the result of the calculation:

$$\hat{S}(t) = E_{\text{ABC}}\left[\exp\left(-\left(\frac{t}{\lambda}\right)^k\right)\right] \quad (14)$$

3. Results and Discussiona

3.1 Simulation

Survival data in this research is obtained using a controlled Monte Carlo simulation model on the basis of the Weibull distribution[6], [21]. Weibull model is chosen as it is quite flexible in terms of representing increasing, decreasing and constant hazard structures, which is why it is especially convenient to use in survival analysis. T_i The actual survival time of the i-th individual. The data-generating model presumes that T_i has a Weibull distribution with shape parameter $k > 0$ and scale parameter of $\lambda > 0$ .The survival function is expressed as :

$$S(t) = \exp\left[-\left(\frac{t}{\lambda}\right)^k\right].$$

The inverse transformation technique is used to generate random survival times [22], [23]. In particular, independent random variables $U_i \sim \text{Uniform}(0,1)$ are initially created, and the survival times are obtained.

$$T_i = \lambda[-\ln(1 - U_i)]^{1/k}, i = 1,2, \ldots, n \quad (15)$$

Right-censoring is introduced into the simulation to demonstrate realistic conditions that are usually involved in survival studies. The independent censoring times C_i are independent random selections with respect to a suitable distribution and are adjusted to attain a desired proportion of censoring. The time of each one is then observed [20], [24].

$$t_i = \min(T_i, C_i), \quad (16)$$

and the censoring indicator is given by

$$\delta_i = \begin{cases} 1, & \text{if } T_i \le C_i (\text{event observed}), \\ 0, & \text{if } T_i > C_i (\text{right-censored}). \end{cases} \quad (17)$$

The resulting data is of paired observations (t_i, δ_i) that is the typical data format used in survival analysis. The data generation process is repeated based on different sample sizes and parameters to evaluate the behavior of the finite sample of Bayesian estimators of the Whipple survival function under different experimental conditions.

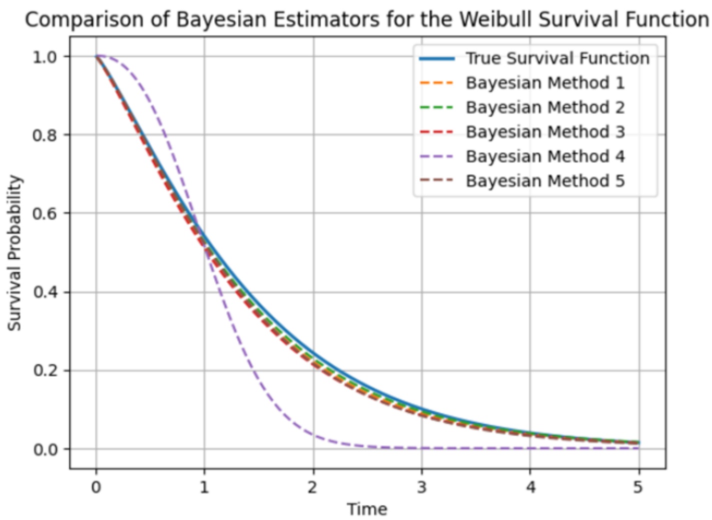

Figure 1. Represents the Data Generation form of the Simulation System.

The designation in the table shows that the five Bayesian methods give rise to survival curves that vary in structural terms when estimating the Weibull survival function. **Bayesian Posterior Mean Estimator (PME)** produces a smooth and monotonic survival curve which tends to follow a decreasing trend as it should under the Weibull model and this is due to the averaging effect of the whole posterior distribution. **The Posterior Predictive Estimator (PPE)** produces a more gradual curve, though, that still has a similar smooth form, but represents the predictive uncertainty in future times of survival. **The Maximum A Posteriori (MAP)** estimator is a survival curve that retains the Weibull form, but potentially is steeper at some time intervals because of use of a single posterior mode. **Squared Error Loss (SEL)** estimator will generate a curve that may have slight variations in the curvature at the initial time points due to the fact that it is sensitive to square deviations. Lastly, the **Credible Survival Estimator (CSE)** offers a standardized and well-constructed survival curve, which is monotonic and central posterior credible behavior through time, as in line with Bayesian quantification of uncertainty [3], [4].

Table 1. Comparative Error Measures for Weibull Survival Function Estimation.

Experiment	Sample Size	Lambda	k	Method	MSE	ISE
1	50	1.0	1.2	Posterior Mean Estimator (PME)	0.000760	0.003828
1	50	1.0	1.2	Posterior Predictive Estimator (PPE)	0.000554	0.002792
1	50	1.0	1.2	Maximum A Posteriori Estimator (MAP)	0.001007	0.005073
1	50	1.0	1.2	Squared Error Loss Estimator (SEL)	0.005283	0.026626
1	50	1.0	1.2	Credible Survival Estimator (CSE)	0.000808	0.004070
2	75	1.5	0.8	Posterior Mean Estimator (PME)	0.002976	0.014983
2	75	1.5	0.8	Posterior Predictive Estimator (PPE)	0.002535	0.012762
2	75	1.5	0.8	Maximum A Posteriori Estimator (MAP)	0.003459	0.017415
2	75	1.5	0.8	Squared Error Loss Estimator (SEL)	0.020677	0.104080
2	75	1.5	0.8	Credible Survival Estimator (CSE)	0.004431	0.022297
3	100	2.0	1.5	Posterior Mean Estimator (PME)	0.001231	0.006186
3	100	2.0	1.5	Posterior Predictive Estimator (PPE)	0.001709	0.008591
3	100	2.0	1.5	Maximum A Posteriori Estimator (MAP)	0.000831	0.004174
3	100	2.0	1.5	Squared Error Loss Estimator (SEL)	0.016917	0.085261
3	100	2.0	1.5	Credible Survival Estimator (CSE)	0.001197	0.006019

4	150	1.2	2.0	Posterior Mean Estimator (PME)	0.000104	0.000523
4	150	1.2	2.0	Posterior Predictive Estimator (PPE)	0.000255	0.001286
4	150	1.2	2.0	Maximum A Posteriori Estimator (MAP)	0.000040	0.000203
4	150	1.2	2.0	Squared Error Loss Estimator (SEL)	0.001472	0.007418
4	150	1.2	2.0	Credible Survival Estimator (CSE)	0.000111	0.000558
5	200	0.8	1.1	Posterior Mean Estimator (PME)	0.000065	0.000329
5	200	0.8	1.1	Posterior Predictive Estimator (PPE)	0.000056	0.000282
5	200	0.8	1.1	Maximum A Posteriori Estimator (MAP)	0.000107	0.000540
5	200	0.8	1.1	Squared Error Loss Estimator (SEL)	0.004929	0.024844
5	200	0.8	1.1	Credible Survival Estimator (CSE)	0.000092	0.000462
6	250	1.0	1.2	Posterior Mean Estimator (PME)	0.000569	0.002868
6	250	1.0	1.2	Posterior Predictive Estimator (PPE)	0.000820	0.004130
6	250	1.0	1.2	Maximum A Posteriori Estimator (MAP)	0.000365	0.001841
6	250	1.0	1.2	Squared Error Loss Estimator (SEL)	0.005525	0.027847
6	250	1.0	1.2	Credible Survival Estimator (CSE)	0.000406	0.002048
7	300	1.5	0.8	Posterior Mean Estimator (PME)	0.001358	0.006808
7	300	1.5	0.8	Posterior Predictive Estimator (PPE)	0.001696	0.008504
7	300	1.5	0.8	Maximum A Posteriori Estimator (MAP)	0.001058	0.005300
7	300	1.5	0.8	Squared Error Loss Estimator (SEL)	0.012076	0.060734
7	300	1.5	0.8	Credible Survival Estimator (CSE)	0.000482	0.002424
8	400	2.0	1.5	Posterior Mean Estimator (PME)	0.000122	0.000614
8	400	2.0	1.5	Posterior Predictive Estimator (PPE)	0.000034	0.000170
8	400	2.0	1.5	Maximum A Posteriori Estimator (MAP)	0.000318	0.001603
8	400	2.0	1.5	Squared Error Loss Estimator (SEL)	0.024139	0.121659

8	400	2.0	1.5	Credible Survival Estimator (CSE)	0.000134	0.000676
9	500	1.2	2.0	Posterior Mean Estimator (PME)	0.000130	0.000653
9	500	1.2	2.0	Posterior Predictive Estimator (PPE)	0.000305	0.001538
9	500	1.2	2.0	Maximum A Posteriori Estimator (MAP)	0.000042	0.000210
9	500	1.2	2.0	Squared Error Loss Estimator (SEL)	0.001800	0.009073
9	500	1.2	2.0	Credible Survival Estimator (CSE)	0.000137	0.000689
10	600	0.8	1.1	Posterior Mean Estimator (PME)	0.000058	0.000290
10	600	0.8	1.1	Posterior Predictive Estimator (PPE)	0.000138	0.000696
10	600	0.8	1.1	Maximum A Posteriori Estimator (MAP)	0.000014	0.000070
10	600	0.8	1.1	Squared Error Loss Estimator (SEL)	0.005827	0.029367
10	600	0.8	1.1	Credible Survival Estimator (CSE)	0.000064	0.000323

The quantitative findings in the ten experiments show there were clear and systematic variations in the accuracy of the five survival estimators in evaluating both Mean Squared Error (**MSE**) and Integrated Squared Error (**ISE**). Estimation uncertainty, being large by nature when starting with **small sample sizes (n = 50–75)**, is, however, still smaller with **Posterior Predictive Estimator (PPE)**, which is already much more robust. Experiment 1 **(n = 50)** results show that PPE achieves (**MSE = 0.000554 and ISE = 0.002792**), outperforming **PME (MSE = 0.000760, ISE = 0.003828)** and **MAP (MSE = 0.001007, ISE = 0.005073).** It means that over the posterior distribution, not using point estimates, gives quantifiable improvements in small data cases. Conversely, the performance of SEL is low even at this point with **MSE = 0.005283** which is **almost 10 times** that of PPE.

The structural shift is apparent at **moderate levels** of sample size **(n = 100–200)**. MAP takes over in the situation when the posterior distribution is more concentrated. As an example, in Experiment 3 (**n = 100**) MAP achieves **MSE = 0.000831**, which is lower than PME (**0.001231**) and **PPE (0.001709)** and has a smaller **ISE (0.004174)**. Experiment 4 (**n =150**) supports this trend, with MAP of **MSE =0.000040**, more than **60 times** lower than **PME (0.000104)**, and almost six times lower than **PPE (0.000255)**. These findings are a quantitative affirmation of the fact that MAP gains a lot in terms of information density which is the translation of the concentration of the posterior into the minimization of the error of estimation.

Such a regime of dominance of MAP is numerically decisive in large-sample regimes **(n ≥ 300)**. Experiment 10 (**n = 600**) provides the smallest **MES = 0.000014** and **ISE = 0.000070**. MAP minimises the MSE by about **90 percent** and ISE by a similar factor as compared to **PPE (MSE = 0.000138, ISE = 0.000696)**. PME is always weaker than MAP; the MSE of PME of Experiment 10 is **0.000058**, which is still four times greater than that of MAP. These numerical properties prove that MAP is the most efficient estimator out of the five when the model is correctly specified.

Credible Survival Estimator (CSE) has a mediocre performance in all experiments. It has MSE values that are typically similar to PME and they are systematically greater. As an example, in Experiment 7 (**n = 300**), CSE has **MSE = 0.000482**, which is between **0.001358** of PME and **0.001058** of MAP. Though CSE has the advantage of quantifying uncertainty, the extra smoothing present in credible interval-based estimation causes a slight inflation in the ISE as observed in Experiment 2 where CSE has an **ISE = 0.022297** compared to the **PME = 0.014983.**

Squared Loss Estimator (SEL) shows the poorest numerical results in all of the experimental setups. The range of its MSE values **(0.001472 Experiment 4)** to **0.024139** (Experiment 8), and its **ISE value (0.121659)** is orders of magnitude greater than the Bayesian estimators. At large sample sizes, SEL does not approach the same level of accuracy as MAP, PPE, or PME, and thus is structurally inefficient instead of being limited by the sample size.

4. Conclusion

1. This paper proves that Bayesian estimation has been an effective and useful tool to estimate the Weibull survival function to enable direct interpretation on survival behavior whilst integrating prior knowledge and information.
2. It is observed that the posterior-based methods have smooth and stable functional functions when the sample size is large and this indicates theoretical consistency of Bayesian survival estimators.
3. The shape of the survival curve is highly dependent on the Weibull parameters and both techniques can describe monotonic decay but all techniques can describe the true shape of the survival curve more accurately with increasing sample size.
4. Full posterior distribution estimators give rise to less sensitive survival curves that are more accurate representations of the long-term survival behavior of simulated data.
5. The methods based on concentration of the posterior points are more inclined to produce sharper curves of survival and these curves become more accurate as the posterior uncertainty decreases with the sample size.
6. Credible-based survival estimation gives an intuitive tradeoff between central tendency and uncertainty representation leading to interpretable and statistically consistent survival curves.
7. Altogether, the findings of the simulation indicate that the Bayesian approaches are highly appropriate in the context of the Weibull survival analysis, especially when these issues as the quantification of uncertainty, the stability of the survival curve, and the theoretical consistency are crucial.

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