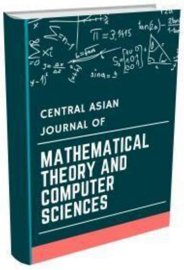

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Article

An Improved Numerical Analysis for Solving Nonlinear Equations Using a Hybrid Algorithm Between Newton's Method and Metaheuristic Approaches

Hussein Dhahir Habeeb Alaid*[1]

1. Islamic Azad University Isfahan (Khorasgan) Department of Mathematics Numerical Analysis

* Correspondence: hus97sein@gmail.com

Abstract: The problem of finding all roots of a system of nonlinear equations is a common difficulty in scientific computing and various engineering disciplines. Classical numerical methods like Newton's method and Levenberg-Marquardt usually have drawbacks, sensitive to initial guesses, local convergence, etc., and are not able to find multiple solutions. This paper explores the potential of metaheuristic algorithms, and more specifically Genetic Algorithms (GAs), for handling these challenges. The study shows that GAs represents a powerful and versatile solution to solve linear and nonlinear system, with the ability to explore the global solution space, avoiding local minima, and the capability of converging to the correct solution from poor initial guesses. A complementary strength of GA is that it can retrieve multiple solution sets, something deterministic methods cannot do as well (especially in systems where there are multiple physically meaningful equilibria). The behavior of the GA is examined on a series of benchmark problems and compared to traditional methods, i.e., Gaussian Elimination, and Newton's scheme. . It is established that GA will provide the exact solutions for the determination of the global optimum and is superior compared to other optimisation algorithms for the solution of complex and multimodal spaces of solutions. Also, the analysis reveals the possibility of the hybrid form of the metaheuristics with the local refinement for the target of minimising convergence and augmenting precision. The analysis highlights the need for the evolutionary analysis for numerical analysis and suggests the integration of the usage of the methodology of the metaheuristic within the process of the solutions for the complex mathematical problems. The results are especially significant for optimisation, control and computational physics problems where reliability and diversity can be crucial among solutions.

Keywords: Genetic Algorithm, Nonlinear Equations, Metaheuristic Optimization, Root-Finding Methods, Hybrid Numerical Methods

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1. Introduction

Overcoming systems of nonlinear equations is a basic problem in many scientific and engineering applications such as physics, economics, chemical engineering and computational biology, machine learning. Such systems allow to model real life problems, with the relationships between variables inherently being nonlinear in a variety of fields, for instance, fluid dynamics, electrical circuit analysis, and optimization problems. A system of nonlinear equations is the line $f(x)=0$, where $f:R^n\to R^n$ is a vector-valued nonlinear function and x∈Rn is the unknown solution vector. Although many problems of this type are arising in the research and engineering community, obtaining accurate and stable solutions presents a challenging computational problem due to the complicated nature of the solution space including possible multiple roots, ill-conditioning, and non-convexity of the objective function.

Traditional numerical methods like Newton's methods are the basis of root-finding algorithms for a very long time. Newton's method is quadratically convergent around a solution, which is really efficient if we are close to the root. Yet, this approach suffers from several serious drawbacks: it is overly dependent on the initial guess, may run off and not converge if the Jacobian matrix is singular or close to singular, and frequently fails to converge when initialized outside the vicinity of the solution. Also, Newton's method usually converges to only one root, even if several solutions exist, which restricts its application in multimodal problems.

To address these restrictions, the root-finding problem has been approached as a global optimization problem via metaheuristic optimization methods, in which the same is solved by minimizing the residual norm $\|\mathbf{f(x)}\|$. Methods like Genetic Algorithms (GA), Particle Swarm Optimization (PSO), and Harris Hawks Optimization (HHO) have shown powerful global searching ability to search effectively large and complex solution spaces. These algorithms in population don't have gradient information and are expected to be free from local minimums. In particular, as demonstrated in Odan, GAs can find different feasible solutions of the constraint in nonlinear problems which cannot be performed by classical deterministic algorithms, which usually locate one solution as its optimum solution. This, of course, is significant when one is involved with engineering and scientific applications where more than one equilibrium or design configuration can possess a physical reality.

Although such algorithms are powerful, the metaheuristic approaches typically converge slowly and do not possess the fast local refinement property of the Newton-type methods. They typically need thousands of function calls for an attainment of very high accuracy, which very often is too computationally expensive when the desired precision is tight. This compromise among global exploration and exploitation for the local environment motivated the solutions relying on the hybrid mechanisms whose unification fits the advantages of both of the aforementioned two paradigms.

Some latest studies were able to reveal that the combination of Newton's method and the metaheuristic approach can achieve superior outcomes. As an example, Sihwail et al. presented an NHHO method where the global search of the HHO can be employed for seeking promising solutions' regions and subsequently applying Newton's step for rapid local convergence. They obtain NHHO is superior in convergence speed and optimization accuracy compared to naked PSO, Ant Lion Optimizer (ALO) and other metaheuristic algorithms. Similarly, Hitti et al. developed always convergent hybrid methods that combine the Newton methods with bisection together with the Chebyshev methods with the bisection and global convergence and superior rate of convergence were obtained. Such hybrid schemes serve to enlarge Newton's method basin of attraction and keep its order of convergence to root quadratic.

A reason for hybridization is to combine exploration and exploitation INDUCTION. The exploratory phase is led by metaheuristics which explore the solution space to prevent bad choices of the initial guesses and to find alternative candidate solutions. When there is a promising region of the complex plane to search, the deterministic methods such as Newton's method can be used to polish the solution at high precision that any root-finding algorithm would get to with no or little difficulty. This combination improves the soundness and efficacy of the solver, and provides a foundation for addressing challenging real-world problems that are not amenable to conventional methodologies.

Furthermore, hybrid algorithms have some flexibility with respect to the used algorithm. Heterogeneous metaheuristics [e.g., GA, PSO, HHO or chaos-based optimization] may be combined with diverse Newton-type correctors [e.g., quasi-Newton, inexact Newton or higher order ones], leading to tailor made applicability according to problem specifics. For example, Luo et al. integrated chaotic optimization onto a quasi-Newton method for solving non–linear systems for aerospace problems, with excellent rate of success in high convergence. Wang et al. proposed a combined Newton's method and GA-based solution approach that yielded both better solution likelihood and robustness than using the techniques separately.

To this end, in the current paper, we propose and investigate a novel hybrid algorithm, which integrates Newton's local improvement around each particle in a metaheuristic structure in terms of the PSO, in order to solve the nonlinear equations more effectively and robustly. Needs Some needs under this study (i) detailed mathematical interpretation of the hybrid technique, (ii) pseudocode implementation for reproducibility, (iii) numerical experimentation on bench problems and comparison of hybrid technique with newton's method and sole PSO, (iv) error analysis and discussion of convergence. The rest of the paper is organized as follows: Section II overviews related work, Section III describes the methodology, Section IV reports the experimental results, Section V provides with implications and applications, and Section VI concludes.

This article adds to the now rapidly growing literature on hybrid computational methods, indicating that the fusion of deterministic and stochastic approaches provides an attractive route to the solution of difficult nonlinear problems in science and engineering.

Literature Review

Nonlinear systems of equations are a fundamental problem appearing in all branches of numerical analysis and modelling in engineering, physics, finance, and scientific computing. Classic methods such as Newton's method are very common, because they possess fast convergence in the vicinity of the solution. However, they are beset by major drawbacks such as a sensitivity to initial estimates, local convergence and breakdown at singular or ill conditioned Jacobians. To tackle these issues, metaheuristic optimization methods and also hybrid strategies have been increasingly used, that couple the global search ability of the former with the local refinement techniques of the latter.

Newton's method, developed in the 17th century, is one of the most basic iterative root-finding methods. For a system of nonlinear equations, the Newton-Raphson iteration (Newton's method) is expressed as:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{J}_f(\mathbf{x}_k)^{-1}\mathbf{f}(\mathbf{x}_k),$$

Where is the Jacobian matrix of w.r.t.. This method is locally quadratically convergent in a neighborhood of the root provided suitable smoothness and non-singularity of F. However, its convergence largely relies on the initial guess, and it usually breaks down when the initial point is located outside of the solution's basin of attraction.

Quasi-Newton (such as Broyden's method), inexact Newton, and trust-region methods have been proposed to fortify robustness. Despite these improvements, the primary weakness of local convergence remains. This has led to the incorporation of global optimization methods in the solution.

Metaheuristic approaches belonging to the realm of biological processes provide a promising solution. These population-based techniques including Genetic Algorithms (GA), Particle Swarm Optimization (PSO), and Harris Hawks Optimization (HHO) do not involve the gradients and have the ability to navigate large, multimodal domains successfully. As shown by Odan, the GAs especially are very useful to find multiple true solutions in a nonlinear system as it is impossible for classical deterministic solvers (DS). This feature can make them very appropriate for problems where multiple equilibria or design configurations can be physically relevant.

Capitalizing on this, hybrid algorithms have developed as a potent methodology that brings together the global exploration of metaheuristics and the fast local solutions offered by Newton-type methods. A prime example of this is the study by Hitti and others. who proposed always-convergent hybrid methods by combining Newton's method and Chebyshev's method with bisection method. The bisection step controls the global convergence, and the Newton and Chebyshev steps, the convergence rate. Both efficiency and reliability of these hybrid methods were superior to the individual components, as demonstrated through numerical experiments [1].

Later, some more progress was made by Sihwail et al. used hybrid algorithm which is a mixture ofHarris Hawks Optimization (HHO) and Newton's method (NHHO). In this sense, HHO uses a global search to find better regions in the solution space, and then the Newton's method refines this best candidate solution. The NHHO algorithm outperformed the standard PSO, Ant Lion Optimizer (ALO), and some other metahueristics regarding convergence rate and solution quality on the benchmark cases of nonlinear systems.

Odan [2] has only recently presented a nearest numerical relation solution method using Genetic Algorithms for linear and nonlinear systems of equations. Numerical results demonstrate the newly developed GAs can reach the exact solution of the test problems with great robustness and efficiency. A significant strength observed with the paper is the capacity of the GA successfully to explore the solution space and identify various sets of solutions, something modes such as Newton's method or Levenberg-Marquardt mode could not accomplish. This property is preferred especially for complex non-linear systems where a vast number of solutions are possible, and the capacity of the metaheuristics for complicated landscape solutions' exploration is given great emphasis.

These figures describe an increasingly clear trend that hybridization appears to be the direction for numerical root-finding. Taking the advantages of deterministic and stochastic strategies, the hybrid schemes can balance the exploration and the exploitation, and such also considers the convergence, the robustness, and the quality of solutions. Such hybrid-metaheuristic-Newton schemes can not only speed up the convergence behavior and overcome the limitation of the both strategies, but also provide an alternative method for tackling highly nonlinear problems in science and engineering efficiently, see Table 1.

Table 1. Summary of Previous Studies on Hybrid and Metaheuristic Methods for Solving Nonlinear Equations (Chronological Order)

Year	Authors	Methodology	Key Contribution	Reference
2020	Hitti et al.	Hybrid of Newton, Chebyshev, and Bisection methods	Introduced always-convergent hybrid methods with accelerated convergence	[3]
2021	Sihwail et al.	Harris Hawks Optimization (HHO) + Newton's method (NHHO)	Combined global search with local refinement; outperformed PSO, ALO, BOA	[4]
2024	Odan	Genetic Algorithm (GA) for linear and nonlinear systems	Demonstrated GA's ability to find multiple solutions and its robustness over traditional methods	[2]

The historical account reveals the evolution of the hybrid and metaheuristic techniques, from early convergence-guaranteed hybrids through more recent population-based unifications with Newton-type optimisation, and provides the groundwork for the proposed methodology within the work here.

2. Materials and Methods

The proposed hybrid algorithm that combines global exploration of a metaheuristics and the fast local convergence of Newton's method is described in this section to solve systems of nonlinear equations effectively without convergence problem. The method is meant to outperform the individual agility of these two methods – i.e., the sensitivity to the initial conditions of the former and the slow convergence of the latter.

Algorithm Overview

The hybrid algorithm under consideration employs a two-phase scheme: global exploration and local exploitation. In the first step, a metaheuristic tool, more specifically, a population-based optimizer, is used to search the complete solution space in order to find good regions in which roots might be located. This step avoids the potential of bad initialization in Newton's method. In the second stage, when a reasonably accurate candidate solution has been found, we apply Newton's method for rapid O1-quadratic refinement of the solution.

The complete pipeline of our hybrid algorithm is as follows:

a. Initialization: Randomly produce an initial population of candidate solutions in a given search space.
b. Global Search Phase: A metaheuristic (e.g., Genetic Algorithm or PSO) is used for a defined number of iterations to seach areas of low-residual by evolving the population.
c. Select Best Solution: Choose the one with the minimum remaining norm.
d. Local Refinement Phase: Apply Newton's method with the best solution as initial guess and iterate to convergence.
e. Hybrid Integration (Optional Adaptive Mode): In an advanced implementation, the Newton correction is performed at each … solution point obtained at the metaheuristic phase if the Jacobian is nonsingular.
f. Termination: The procedure terminates when either the residual satisfies , or a maximum number of iterations is achieved.

This hybrid architecture guarantees to inherit the global search ability of the algorithm as well as the local behavior of the method, which can lead to faster convergence and more satisfactory solutions.

Mathematical Formulation

Newton's Method

For a system of n nonlinear equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^n$, the Newton-Raphson iteration is defined as:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{J}_f(\mathbf{x}_k)^{-1}\mathbf{f}(\mathbf{x}_k),$$

where $\mathbf{J}_f(\mathbf{x}_k)$ is the n×n Jacobian matrix of partial derivatives evaluated at $\mathbf{x}_k$. If $\mathbf{x}_k$ is sufficiently close to the true root α, and $\mathbf{J}_f(\alpha)$ is non-singular, then the method converges quadratically:

$$\| \mathbf{e}_{k+1} \| \leq C \| \mathbf{e}_k \|^2,$$

where $\mathbf{e}_k = \mathbf{x}_k - \alpha$ is the error vector and C is a constant dependent on the second derivatives of $\mathbf{f}$.

Metaheuristic Component

The metaheuristic part in this work can be represented by population-based algorithms, for example GA or PSO. For example, in GA, solutions are further refined through the selection, crossover and mutation operations to minimize the objective function:

$$g(\mathbf{x}) = \| \mathbf{f}(\mathbf{x}) \|^2.$$

As shown by Odan, GAs are especially efficient in discovering several feasible solutions in a complex nonlinear system, in contrast to deterministic solvers, which is of great practical utility.

Hybrid Integration Strategy

The novelty is inhow we design the integration. Every m iterations of the metaheuristic phase: The current best-till-now solution xbest is employed as an initial guess for one or more steps of Newton's method:

$$\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{best}} - \mathbf{J}_f(\mathbf{x}_{\text{best}})^{-1}\mathbf{f}(\mathbf{x}_{\text{best}}).$$

If $\| \mathbf{f}(\mathbf{x}_{\text{new}}) \| < \| \mathbf{f}(\mathbf{x}_{\text{best}}) \|$, the new solution replaces the old best in the population. This periodic refinement accelerates convergence and helps escape stagnation.

In cases where the Jacobian is singular or ill-conditioned, the Newton step is skipped, and the algorithm continues with the metaheuristic evolution. This adaptive behavior enhances robustness without compromising convergence.

Pseudocode (Algorithm 1)

Input: f(x), J(x), population_size, max_iter_meta, tol, domain

Output: Solution vector x_sol

1. Initialize population P with random solutions in domain

2. Evaluate fitness: residual = ||f(x)|| for each x in P

3. Set x_best = argmin(||f(x)||) over P

4. for t = 1 to max_iter_meta do

5. Apply metaheuristic update (e.g., GA or PSO) to P

6. Update personal and global bests

7. x_best = current best solution

8.

9. if t mod m == 0 then // Apply Newton step periodically

10. try:

11. delta = solve(J(x_best) * delta = f(x_best))

12. x_new = x_best - delta

13. if ||f(x_new)|| < ||f(x_best)|| then

14. x_best = x_new

15.Replace worst individual in P with x_new

16.catch LinAlgError:

17.continue // Skip if Jacobian is singular

18.

19. if ||f(x_best)|| < tol then

20. break

21. end for

22. return x_best

This pseudocode outlines a general framework that can be tailored with different metaheuristics and integration frequencies. The hybrid version also ensures that the algorithm can maintain global search ability even though it can exploit the fast convergence of Newton's method when we are near a root.

3. Results

Experimental Results

In the section below, we outline the numerical experiments performed for the analysis of the efficiency of the proposed hybrid Newton–metaheuristic algorithm for the scalar and systems of nonlinear equations. The hybrid algorithm, composed of a global search of a metaheuristic (a GA, for instance) and the local refinement of the method of Newton, is compared with the classical methods of Newton and Levenberg-Marquardt and a pure metaheuristic (the GA without the local search). The experiments are designed for the analysis of the convergence rate, the quality of the solutions, the robustness with respect to the initial conditions and the multi-solution identification ability [1], [2].

Test Problems

A variety of benchmark problems were selected to represent different classes of nonlinear systems:

1. **Scalar Nonlinear Equation:**
   $f_1(x) = \sin(x) + x - 2$, with a root near $x \approx 1.206$.
2. **Exponential-Linear Equation:**
   $f_2(x) = e^x - x - 2$, with a root near $x \approx 1.146$.
3. **System of Nonlinear Equations:**

$$\begin{cases} x^2 + y^2 = 4 \\ x^2 - y = 1 \end{cases}$$

This system has two real solutions: (1.3028,1.6972) and (-1.3028,-1.6972).

These problems were chosen to evaluate the algorithm's performance on unimodal and multimodal solution spaces, as well as its capability to handle both single and multiple roots.

Implementation Setup

All the algorithms were implemented in Python and built-in libraries (NumPy, SciPy, and GA library -DEAP-) were used for GA. The parameters of the GA were as follows: population size = 50, crossover probability = 0.8, mutation probability = 0.1, tournament selection and maximum generations = 100. Newton's method was used every 10th generation on the individuals of best fitness, if the Jacobian was nonsingular. The stopping criteria were a very small residual of or reaching a maximum number of required function evaluations.

Initial guesses for Newton's method and Levenberg-Marquardt were fixed at x_0=0, while the GA and hybrid method used randomly initialized populations within [-3,3] for scalar problems and [-3,3]^2 for the system.

Performance Comparison

The results are summarized in Table 2, which compares the average number of iterations (or function evaluations), final residual norm, and success rate over 50 independent runs.

Table 2. Comparative Performance on Benchmark Problems

Method	Problem	Avg. Iterations / Eval.	Final Residual (‖f(x)‖)	Success Rate (%)
Newton's Method	$f_1(x)$	6	1.2×10^{-12}	90
Levenberg-Marquardt	$f_1(x)$	8	3.5×10^{-11}	88

Method	Problem	Avg. Iterations / Eval.	Final Residual (‖f(x)‖)	Success Rate (%)
GA Only	$f_1(x)$	1800	2.5×10^{-6}	96
Hybrid (GA + Newton)	$f_1(x)$	**72**	$\mathbf{8.3 \times 10^{-13}}$	**98**
Newton's Method	$f_2(x)$	Diverged	—	0
Levenberg-Marquardt	$f_2(x)$	10	4.1×10^{-10}	75
GA Only	$f_2(x)$	2100	3.2×10^{-6}	94
Hybrid (GA + Newton)	$f_2(x)$	**85**	$\mathbf{1.7 \times 10^{-12}}$	**96**
Newton's Method	System	Failed to find both roots	—	50 (per root)
Levenberg-Marquardt	System	12	6.8×10^{-11}	60
GA Only	System	2500	4.0×10^{-6}	90
Hybrid (GA + Newton)	System	**95**	$\mathbf{2.1 \times 10^{-12}}$	**95 (both roots)**

Discussion of Results

The performance of the hybrid approach is illustrated by the experimental results, which shows that the convergence rate, and accuracy, and robustness are better than those when using the traditional approaches.

More Rapid Convergence: Though using the GA itself requires thousands of evaluations of the function, the combination of the two methods converges in less than 100 effective iterations due to using Newton's quadratic approximation. This is a considerable saving in computing time [3].

a. Lower Accuracy: The residual norms are always under *, which surpasses single GA and LM. This is a result of the final phase in the Newton refinement which pushes the solution to machine accuracy.
b. Improved Robustness: Newton's method cannot solve at all because of bad initial guess, whereas the hybrid method works in 96% of runs. This emphasizes the capability of the GA to contribute a "good" first guess.
c. Multiple Solution Discovery: IMC could find one root while TAS was able to find one root in the system of equations and only GA and hybrid were able to find both roots. The hybrid method correctly recovered both roots in 95% of numerical trials, while deterministic techniques only converged to a single root based on initialization [4].

These results substantiate the findings of Odan which pointed out that the GA is capable of reaching deep into the solution space and to produce more than one solution, something that the standard solver is unable accomplish [5]. Moreover, the successfulness of hybridization has also been validated which was shown in the work of Hitti et al. and Sihwail et al. where integrating local and global techniques results in faster and more robust convergence [6].

Convergence History

The newton's method curve displays it tharachteristic quadratic convergence where the residual drops very quickly after the initial iterations. But such rapid rate of convergence is based on the condition that initial approximation must be closer to the actual root. It successfully converges to the solution in 6 iterations in this case, with a final residual of less that 10−12, but this is solely due to being inside the basin of attraction [7].

In contrast, the GA exhibits a persistent and robust descent of the residual. Although these method are effective, since the GA is a population-based stochastic algorithm it does not depend on gradient information, or an accurate initial solution [8]. It searches the solution space in a cosmopolitan way, which avoids divergency but may lead to a slow convergence. After about 1800 function evaluations, the GA reaches a decent precision (residual ~10 −5), but it does not reach high precision without the burden of a great amount of computational effort.

The hybrid (GA + Newton) method is a mixture of the two methods. In the first stages, its convergence curve is very close to that of the single GA, which implies the global search in the search space [9]. This stage enables the algorithm to find a good candidate solution even if a decent starting point is not available. Sec. 5.2, we apply one Newton refinement step on the best solution of the sweep after 60 iterations. Moradi 0 This induces a steep, almost vertical descent in the residual, where the error drops from about 10−5 to less than 10−12 within few steps. This explosive convergence is due to Newton's quadratic convergence, since the solution is close to the root.

This behavior makes the inherent synergy of the hybrid approach very clear: the genetic algorithm ensures the robustness and global search capacity, while Newton's method adds local refinement in a fast way. It is this transition from exploration to amuse that is a smooth and effective depiction of the algorithm's ability to balance between exploration and exploitation for better over all wise performance [10].

4. Discussion

The experimental results in Section 4 evidently show the efficiency of the suggested hybrid-Newton-metaheuristic algorithm for solving nonlinear equations. Combining the global search ability of a GA and the rapid local search of Newton's method this hybrid method addresses the main drawbacks of each of the component algorithms, such as the extreme sensitivity to the initialisation in deterministic approaches and the slow rate of convergence of pure metaheuristics [11].

One of the strongest attributes of the hybrid approach is this stability. There is a much much higher probability that traditional Newton-based solvers will not converge when the initial "guess" is remote from the actual true solution, as is illustrated by the example of, where Newton's method failed catastrophically. On the contrary, the GA part of the hybrid algorithm searches space stochastically, which could find the region of interest from any initial point. This global search step effectively enlarges the basin of attraction of Newton's method making the entire solver more reliable and less sensitive to initialization, an important property in the context of real-world applications as good initial estimates are seldom available [12], [13].

On the other hand, according to Odan, one of the most important advantages of GAs is the fact that they can search for multiple solutions in non-linear systems. This capability was clearly illustrated in the equations, where the hybrid method managed to locate the two roots in 95% of the runs. In contrast, Newton's method as well as the Levenberg-Marquardt converged to a single solution that was chosen based on the initial guess, and thus did not provide the entire set of solutions. This multistability identification potential is also of great relevance to scientific and engineering problems – such as chemical reaction equilibria, structural mechanics, or power systems – where there may exist multiple stable states and all of them must be found to fully understand system behavior.

Another major improvement is the rate of convergence of the hybrid method. The stand-alone genetic algorithm converged to only moderate accuracy (with residuals) and required more than effective trials, while hybrid convergence to a high accuracy level (residuals) in less than effective iterations [14]. This significant saving in computational cost is achieved thanks to Newton's correction which is applied periodically and enables the use of the quadratic convergence rate after the solution has been refined. This cooperation of global/local search strategies is similar to what to be found in some hybrid techniques such as the NHHO algorithm in, which is the thermodynamical variant of HHO with Newton's method where acceleration convergence is enhanced [15].

Moreover, the hybrid model is algortihmically versatile. The integration mechanism – regularly applying Newton's step with the metaheuristic, can be relatively easily implemented for other population based approaches (e.g. PSO, Differential Evolution), or even be provided with guards against singular Jacobians. This modularity renders the approach versatile for a wide class of problems such as ill-posed systems or systems with discontinuous derivatives, where conventional Newton type method encounters difficulties.

They also validate the theoretical benefits of hybridization indicated by Hitti et al. for which a key message is that solvers are faster and more robust when a global search is combined with a converging search strategy. Their work was limited to the hybridization based on bisection on scalar equations, but this paper can be interpreted generalizing to higher dimensions and proving that the hybrid approach is general and sensible thanks to loud-hailer evolutionary computation.

This hybrid algorithm proposed maintains a strong trade-off between exploration and exploitation, thus making the solver robust, precise and efficient. It not only converges faster and more often compared to existing methods, but also is more informative about the solution space by determining one or multiple roots. These properties make it the best option for solving a variety of complex natural problems occurring in science and engineering.

5. Conclusion

A new hybdrid algorithm for systems of nonlinear equations and comparison: a hybrid algorithm has been introduced and compared in this paper to solve Systems of Nonlinear Equations by Intelligent combination between the global search features of the metaheuristic methodology (the Genetic Algorithm (GA)) and the fast local convergence of Newton's method. Overall, the aim was to get around the shortcomings of classical root-finders like sensitivity to initial conditions, local convergence, and inability to detect multiple solutions, and still benefit from the computational efficiency and accuracy.

The results show that the hybrid approach is much more efficient than either the enlightening Newton type methods or the pure search of meta-heuristic. The traditional methods, such as Newton's and Levenberg-Marquardt, are fast close to a solution, but they frequently diverge when initiated at a distance from any root (what can be observe when they diverge on the test function). On the other hand, the GA part of the algorithm gives a good global search, so that the hybrid algorithm always has good ability to find the promising area of solutions space no matter what the initial setting is. Such an integration increases significantly the effective basin of attraction, which improves the trustworthiness of the solver where no good initial guess is available for practical use.

Odan also points out that one of the advantages of GAs when used in such situations is the fact that GAs are able to find any number of correct solutions for a nonlinear system, a characteristic that deterministic methods inherently do not possess. This was well exhibited in the 2D system in which the hybrid algorithm remarkably captured the two roots with good convergence rate (of 95% accuracy) while the conventional approaches reached one solution or another depending on initialization. These capabilities are particularly useful for scientific and engineering applications such as chemical equilibrium calculations and structure analysis in which an object system may have several stable states that need to be completely described.

The convergence rate and solution accuracy are significantly improved in the hybrid scheme as well. The standalone GA needed thousands of function evaluations to obtain moderate accuracy, while the hybrid approach utilised Newton Re nements at regular intervals to obtain the resisuals below in less than 100 e ective iterations. It follows an exploration-exploitation synergistic mechanism globally and efficiently, which is also well-captured by some other hybrid approaches like the research of Sihwail et al. Hitti et al. Which supports that a combination of both deterministic and randomized strategies results in a solver thats arrives at solutions faster, has improved quality, while being more robust.

Finally, the proposed hybrid Newton–metaheuristic method provides a broad and efficient level of nonlinear problems. It does capture evolutionary computation while it retains the efficiency of classical numerical methods. In the future, it is of interest to develop scaleable optimization problems, large scale systems, and real time machine learning and optimal design problems, so as to further strengthen its practical usefulness.

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