

Article

Comparative Study of Padé Approximation Methods in Solving Lane-Emden Type Differential Equations

Hayder Ali Abdulsada Alkinani*¹

1. University of Mohaghegh Ardabili Faculty of Mathematical Sciences Department of Mathematics and Applications

* Correspondence: : haiderasif001@gmail.com

Abstract: This study examines how effective Padé approximation methods are for solving Lane-Emden type differential equations. These are special types of equations that are often used in astronomy, like when understanding the structure of some stars and how gases act in round shapes. These equations are hard because there's an issue at the beginning point and they get more complicated due to the polytropic index. Traditional power series methods usually don't work well beyond a small range because they are only effective within specific limits. Padé approximants are a kind of math tool that can give more accurate estimates. They are effective because they can be used in many different situations and show what happens accurately at places where functions act strangely. In this study, we apply different Padé approximations of different levels to the Lane-Emden equation. They are evaluated based on how correct they are, how fast they find a solution, and how well they use computer resources. Computer simulations are performed, and the results are compared with older techniques, such as the Runge-Kutta method and standard series solutions. The comparison shows that Padé approximants are better because they give accurate results faster over larger areas, especially when the polytropic index goes up. But there are still issues with picking the best orders for the estimates and handling more complicated nonlinear problems. The results show that Padé approximation is a helpful way to understand things while being efficient with computers. This makes it a helpful tool in both ideas and real life, especially when regular methods don't work well due to unusual situations or high computer expenses.

Citation: Alkinani, H. A. A. Comparative Study of Padé Approximation Methods in Solving Lane-Emden Type Differential Equations. Central Asian Journal of Mathematical Theory and Computer Sciences 2025, 6(3), 494-505.

Keywords: Lane-Emden Equation, Padé Approximation, Nonlinear Differential Equations, Polytropic Models, Numerical Analysis

Received: 23rd Apr 2025

Revised: 30th Apr 2025

Accepted: 8th May 2025

Published: 19th May 2025



Copyright: © 2025 by the authors. Submitted for open access publication under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>)

1. Introduction

The Lane-Emden equation is a well-known math formula that shows how things change over time. It is often used in space science and other fields of physics. It is very important to understand how polytropic stars, white dwarfs, and gas spheres with a constant temperature are built inside. This equation allows scientists to analyze the behavior of cohesive spherical fluids when they are steady and in equilibrium. But solving the Lane-Emden equation is not easy because it has a problem at the starting point and includes complicated terms. In some special situations, the Lane-Emden equation has exact solutions that can be calculated. These few special cases, which usually relate to certain values of the polytropic index, are used as standards to check how well numerical and approximation methods work. For general values, it's not possible to find exact solutions, so researchers use numerical methods or approximations to get helpful results. Power series solutions can be used close to the starting point, but they have a limited range

of accuracy and might not provide good results everywhere else. To get around these issues, simpler methods like the Padé approximation have become very popular. Padé approximants are a way to change a power series into a fraction made up of two polynomials: one on top (numerator) and one on the bottom (denominator)[1]. This change allows the approximation to be effective even in areas where a regular series solution isn't correct. Padé approximants often handle strange behavior better and provide more accurate results, especially near important points. In recent studies, scientists have been examining the role of various types of Padé approximations in simplifying the solution process for the Lane-Emden equation. The goal is to see if this method can give better results, faster answers, and easier math than older methods like the Runge-Kutta method or basic power series. This study explores different ways to use Padé approximations to solve a kind of math equation known as Lane-Emden differential equations. It examines how well different levels of Padé approximants perform and compares their results to those from other methods. We check how quickly it works, how correct it is, and how well it uses computer power. The study includes images and numbers to compare known solutions and to see how effective the Padé method is compared to the best results. This research highlights how important it is to pick the correct level of the Padé approximant based on the details of the equation. Simple approximations are user-friendly but tend to lack precision, whereas more complex methods can provide greater accuracy, albeit with increased difficulty in application. Finding the right balance is important for real-world use, especially in computer astrophysics, where both speed and accuracy are needed. In simple terms, this study wants to explain how Padé approximation methods can help us solve Lane-Emden type equations. It examines various examples to assist researchers who want to use semi-analytical methods for challenging nonlinear issues in science and engineering.

2. Materials and Methods

Overview of Padé Approximation

Padé guess could be a valuable math strategy that appears a work as a division made of two polynomials. In differentiate to normal polynomial approximations, such as Taylor arrangement that delineate capacities as unbounded wholes of progressively higher control terms, Padé guess utilizes an elective strategy utilizing divisions. This distinction makes a difference Padé approximants to be much more exact and to work superior, particularly near to focuses where the work has abnormal or complicated behaviour. This strategy is particularly supportive for tackling extraordinary math issues that have complex parts or points where they gotten to be vague, just like the Lane-Emden conditions utilized in space science models[2]. One huge advantage of the Padé estimation is that it can speak to a work more precisely over a bigger extend than a abbreviated Taylor arrangement. A Taylor arrangement works well generally close the point where it starts, but Padé approximants as a rule deliver way better comes about over a more extensive extend. Typically, since the level headed shape can appear capacities in a more flexible way. Having a polynomial within the foot portion of a division makes a difference the approximation appear how shafts and other uncommon focuses influence the work being assessed. Since of this, the Padé estimation is seen as distant an improved way to get it how a work carries on, indeed when it's difficult to portray the work utilizing fair polynomials[3].

Another key highlight of the Padé strategy is that it works well when the Taylor arrangement doesn't meet or takes as well long to focalize. This happens frequently in numerous regular issues, particularly when working with capacities that alter rapidly or have breaks in them Padé guess understands this issue by changing a control arrangement that does not include up well into a less complex division (sound work) that includes up speedier and needs less terms. Typically, an important part of computer programs, where being quick and exact is vital. By utilizing less terms to reach a particular exactness, Padé

approximants make calculations quicker and make numerical strategies more dependable. When fathoming differential conditions, particularly dubious ones just like the Lane-Emden conditions, Pade estimation is exceptionally important. These conditions more often than not have an uncommon point at the root, making it difficult for numerous numerical strategies to work well. Analysts can discover a arrangement by beginning with an arrangement strategy, like utilizing control arrangement, and after that utilizing Pade change. This permits them to form the arrangement work indeed exterior the normal limits[4]. This strategy makes a difference discover near arrangements in zones where customary arrangement strategies do not work well or can't be utilized. So, Pade approximants offer assistance interface easy-to-work-with equations and computer strategies, making them more exact and valuable. Pade estimation is exceptionally valuable, particularly when matched with other comparable strategies. For occasion, when we utilize strategies just like the A domain Decay Strategy, the Homotopy Annoyance Strategy, or the Differential Change Strategy to form an arrangement we can use the Pade approximant on that arrangement. This makes a difference the arrangement get way better and work for a more extensive extend of values. This combination has been exceptionally supportive in tackling troublesome nonlinear conditions. It combines the leading parts of different strategies to make a more grounded and more exact way to fathom issues. These combined methods have ended up prevalent as of late since they are versatile and work well in several zones of material science, building, and applied math. From a hypothetical point of see, the quality of Pade estimation is that it can speak to capacities in ranges where a Taylor arrangement would not work well. It essentially amplifies the work so that it can be caught on way better and more precisely over a bigger extend. This makes the strategy valuable additionally based on strong math. Also, among the different kinds of Pade approximants, the inclining ones where the beat portion (numerator) has the same degree as the foot portion (denominator) usually donate the leading blend of effortlessness and accuracy. These strategies are preferred in physical and building work because they are straightforward and work well[5].

One-Dimensional Differential Transform

The differential transform of the function $y(x)$ is described like this:

$$Y(k) = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0} \quad (1)$$

In math and computer use, especially when solving differential equations with transform methods, a transformed function is very important. In equation (1), $y(x)$ usually stands for the main function related to the problem. This function is based on a real-world situation, like in physics, engineering, or math, and depends on the variable x . This function is usually hard to use directly, especially when it doesn't behave in a straight line or when we try to solve equations mathematically. To make solving easier, we change the original function into a new one called $Y(k)$ [6].

This changed function is usually called the T-function, and it exists in a different area, often using a new variable called k . The main goal of changing the original function $y(x)$ into $Y(k)$ is to make it easier to solve complicated equations. Strategies just like the Differential Change Strategy (DTM) and Laplace Change are commonly utilized for this reason. These methods change the processes of differentiation and integration into simpler math forms that are easier to work with. After finding the solution in the new form, the last step is to get back the original function $y(x)$ [7]. This is done by using the inverse transform method. The differential inverse transform of $Y(k)$ is a method used to get back the original function $y(x)$ from its transformed version. This step is important because we want to show the answer using the original variable x . The inverse transform means adding up a set of values based on the numbers in the transformed function, following certain rules based on the method you use. This method helps to easily and correctly find

the solution, especially for initial value problems and nonlinear ordinary differential equations[8].

$$y(x) = \sum_{k=0}^{\infty} x^k y(x) \quad (2)$$

From (1) and (2), we obtain

$$Y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0} \quad (3)$$

The differential change strategy could be a broadly utilized expository procedure within the field of connected science and designing for tackling differential equations, especially nonlinear standard differential conditions. This strategy depends on the basic thought of changing a differential condition into a set of logarithmic conditions that are less demanding to control and unravel. At the heart of the differential change strategy lies a conceptual association with the Taylor arrangement extension. Particularly, the strategy draws motivation from the Taylor arrangement by developing the arrangement of a work as a entirety of weighted terms including powers of the autonomous variable. In any case, not at all like the classical Taylor arrangement strategy, the differential change approach does not require the typical computation of derivatives at each step, which is frequently a computationally seriously and logically requesting errand[9].

Rather than explicitly calculating higher-order subordinates, the differential change strategy computes them iteratively through repeat relations. These relations are built up by changing the first differential equation into a comparing set of conditions within the change space. This change space is characterized through a set of operational rules that interface the initial work, ordinarily spoken to by lowercase letters, with its transformed counterpart, spoken to by capitalized letters. This notational refinement is used reliably all through the strategy to clearly differentiate between the first work in its physical or time domain and the changed work that exists within the logarithmic or assistant space. For occasion, on the off chance that the first work is denoted by y within the physical space, its changed adaptation within the differential change strategy is spoken to by Y within the changed space. The iterative handle utilized within the differential change strategy is fundamental to its productivity and common sense. Instead of taking progressive subordinates of the first work, the strategy builds each changed term utilizing as of now calculated lower-order terms. This recursive nature makes it especially appropriate for taking care of issues including nonlinear terms, where conventional methods may struggle or require noteworthy typical control[10].

The repeat relations are inferred from the structure of the differential condition itself, and once the changed work is completely set up, the initial work can be reproduced through a reverse change. This reverse prepare includes summing the changed terms increased by powers of the free variable, taking after a structure comparable to a truncated Taylor arrangement. The result is an inexact but often highly exact arrangement to the initial differential condition, communicated in a arrangement shape that meets quickly in numerous viable cases. Additionally, one of the key highlights of the differential change strategy is that it complies with a set of essential scientific operation rules, which are methodically characterized and summarized in operational tables. These rules oversee how changes are connected to scientific operations such as expansion, duplication, and separation. For case, when two capacities are included within the physical space, their comparing changes are too included within the change space. Additionally, when a work is increased by a steady, its change is additionally duplicated by the same consistent. These rules frame the spine of the strategy and are essential for deciphering complex expressions in the original differential conditions into sensible logarithmic relations within the change space[11].

These numerical operation rules are frequently organized, as appeared in what is regularly alluded to as Table 1 in writing. This table incorporates the change rules for fundamental operations such as scalar increase, work expansion, item of capacities, and subordinates of capacities. The presence of such a table altogether streamlines the application of the strategy since it kills the require for clients to determine the change rules each time from scratch. Instep, by alluding to the table, one can methodically apply the fitting change rule to each term within the unique condition and quickly build the changed condition. Within the setting of this consider, the differential change strategy has been utilized to illuminate Lane-Emden sort differential equations, which are known for their particular nature and nonlinear characteristics[12].

By utilizing the lowercase and capitalized documentation for unique and changed capacities, separately, and by leveraging the iterative structure of the strategy, we point to develop exact and solid surmised arrangements. The choice of this strategy is defended by its demonstrated victory in taking care of similar classes of issues, where conventional numerical strategies may experience challenges due to singularities or nonlinearities. Furthermore, the orderly system of the differential change method, supported by the change rules and recurrence relations, offers a clear and organized pathway for inferring arrangements. In outline, the differential transform strategy gives a robust and proficient approach to solving complex differential conditions. Its establishment within the Taylor series extension, combined with an iterative computation of changed terms and upheld by well-defined numerical operation rules, makes it a viable and flexible instrument in numerical modeling. The clarity of documentation and the organized utilize of change tables assist improve its ease of use and consistency, especially in connected areas such as material science, designing, and computational science[13].

Table 1. The elemental operations of one-dimensional diHerential change strategy.

Original function	Transformed function
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)$
$y(x) = cw(x)$	$Y(k) = cW(k)$
$y(x) = dw/dx$	$Y(k) = (k + 1)W(k + 1)$
$y(x) = djw/dxj$	$Y(k) = (k + 1)(k + 2) \cdots (k + j)W(k + j)$
$y(x) = u(x)v(x)$	$Y(k) = ,k U(r)V(k - r)$
$y(x) = xj$	$Y(k) = \delta(k - j) = \begin{cases} 1, & K = j \\ 0, & k \neq j \end{cases}$

In viable applications, particularly in computational arithmetic and building, the first work is frequently not communicated as an unbounded arrangement but or maybe as a limited arrangement estimation. This reflects the impediments of real-world computation, where boundless entireties are not attainable and truncation is fundamental for commonsense purposes. As such, the work is ordinarily spoken to by a limited number of terms that adequately surmised its behavior inside a wanted run of accuracy. In this setting, the initial work, which may depend on a physical variable such as time or space, is approximated by a limited whole of terms including powers of the autonomous variable and the comparing changed coefficients. This representation permits for a proficient and sensible way to analyze and compute arrangements to differential conditions[7].

The choice with respect to how numerous terms to incorporate within the series is basic and is guided by the joining behavior of the arrangement. Particularly, within the scope of this consider, the truncation restrain is decided based on the meeting of the common recurrence. This implies that the arrangement is ended at a point where the extra terms have an immaterial impact on the exactness of the arrangement. The impact of

higher-order terms is considered insignificant when their commitment to the overall work gets to be exceptionally little. This methodology guarantees that the limited estimation holds the basic characteristics of the first work whereas lessening computational complexity. Additionally, when managing with items of capacities within the system of the differential change strategy, the concept of convolution plays a central part[14].

Concurring to the formal definition, in case a work is the item of two unique capacities, each of which can be represented by its individual converse change, at that point the change of their product is computed through a convolution operation. In other words, the item of two capacities within the physical space compares to a convolution of their changed representations within the change space[15].

This rule gives a efficient run the show for dealing with nonlinear terms and intelligent between numerous capacities inside the differential change system. Convolution in this setting includes summing the items of sets of transformed coefficients from each function, where the indices of the sets are orchestrated to entirety up to the same esteem in each term of the result. This strategy captures how the diverse orders of the initial capacities combine to create modern terms within the item work.

The convolution operation is principal for amplifying the differential change strategy to nonlinear differential conditions, where terms including items of capacities regularly emerge. By characterizing this operation clearly, the strategy gets to be more flexible and appropriate to a broader course of issues[16].

$$y(x) = \sum_{k=0}^m x^k y(k) \quad (4)$$

Equation (3) implies that $\sum_{k=m+1}^m x^k y(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

Definition 2.1. If $y(x) = u(x)v(x)$, $u(x) = D^{-1}[U(k)]$, $v(x) = D^{-1}[V(k)]$, and $\otimes$ denote convolution.

Then, $[y(x)] = D[u(x)v(x)] = U(k) \otimes V(k) = \sum_{r=0}^k U(r)V(k-r)$.

3. Results and Discussion

A. Pade' Approximation

Suppose that we are given a power series $\sum_{i=0}^{\infty} a_i x^i$, representing a function $f(x)$, so that

$$f(x) = \sum_{i=0}^{\infty} a_i x^i, \quad (3.1)$$

A Pade' approximation is a rational fraction

$$\left[\frac{L}{M} \right] = \frac{p_0 + p_1 x + \dots + p_L x^L}{q_0 + q_1 x + \dots + q_M x^M} \quad (3.2)$$

In approximation theory, especially when making simple versions of complex functions, a common method used is the Padé approximation. The Padé approximation tries to show a function as a simple fraction made of two polynomial expressions. This estimate is important because it can show key features of a function with fewer parts than a usual power series. An important part of the Padé approximation is that it creates a function that matches the original function's Maclaurin series as closely as it can, based on certain limits. When we talk about the Padé approximation of a function, we look at a fraction where the top part is a polynomial with a degree of L and the bottom part is a polynomial with a degree of M [17].

These two polynomials are selected so that when you divide them, the result is the same as the original function's series expansion for the highest degree possible. In this arrangement, the top part will have L plus one numbers, and the bottom part will have M plus one numbers. There is extra information because both polynomials have a common

factor that doesn't change the value of the rational function when it's applied to both. To make this unclear situation clear and the definition specific, we usually follow a rule: we set the leading number of the denominator polynomial to one. Making the first number in the bottom part of the equation equal to one isn't just for ease; it's important for correctly creating the Padé approximation. By doing this, we set a scale and prevent endless options that come from changing the size of the rational function randomly. So, the total number of unknowns in the system becomes easier to handle. Basically, the approximation includes L plus one separate numbers from the top part and M separate numbers from the bottom part, since the main part has already been set[18].

In total, there are L plus M plus one unknown number that need to be figured out to fully define the rational function. The number of unknown coefficients tells us how accurately the Padé approximant can represent the original power series. With L plus M plus one coefficient, the rational function can usually match the original power series up to the L plus M term[19].

This means that the approximation will match the original function's series expansion starting from the constant term, then the linear term, the quadratic term, and so on, all the way up to the term with the power raised to L plus M . This matching makes sure that the Padé approximation closely follows the original function at the beginning. This is especially useful for approximating functions that have sharp changes or are not simple polynomials, where Taylor expansions wouldn't work well enough. So, deciding to keep the first number in the bottom part fixed and balancing the levels of the top and bottom parts is not random; these choices are essential for building and making the Padé approximation work well. This organized method results in a mathematical function that matches the original series to a certain level. It often provides better accuracy and works better than using simple polynomial equations of the same level[20].

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0 + p_1 x + \dots + P_L x^L}{q_0 + q_1 x + \dots + q_M x^M} \quad (3.3)$$

Multiply both sides of (3.3) by the denominator of right side in (3.3), and compare the coefficients of both sides of (3.3), then we have

$$a_l + \sum_{k=1}^m a_{l-k} q_k = p_l, \quad (l = 0, \dots, M) \quad (3.4)$$

$$a_l + \sum_{k=1}^L a_{l-k} q_k = 0, \quad (l = M + 1, \dots, M + L) \quad (3.5)$$

After solving (3.5)'s linear equation, we obtain q_k ($k = 1, \dots, L$). After entering q_k into (3.4), we obtain p_l ($l = 0, \dots, M$). A $[L/M]$ Padé' approximation that coincides with $\sum_{i=0}^{\infty} a_i x^i$ through order x^{L+M} has thus been built. The Padé' series provides an A-stable formula for an ordinary differential equation if $M < L \leq M + 2$, where M and L are the degrees of the numerator and denominator, respectively[21].

B. Numerical Examples

Example 4.1 (see Table 2 and Figure 1).

$$u''(x) + \frac{2}{x} u'(x) = 2(2x^2 + 3) u(x), \quad 0 \leq x \leq 1, \\ u(0) = 1, \quad u'(0) = 0, \quad (4.1)$$

where $u(x) = e^{x^2}$ is the precise solution. By multiplying (4.1) by x on both sides,

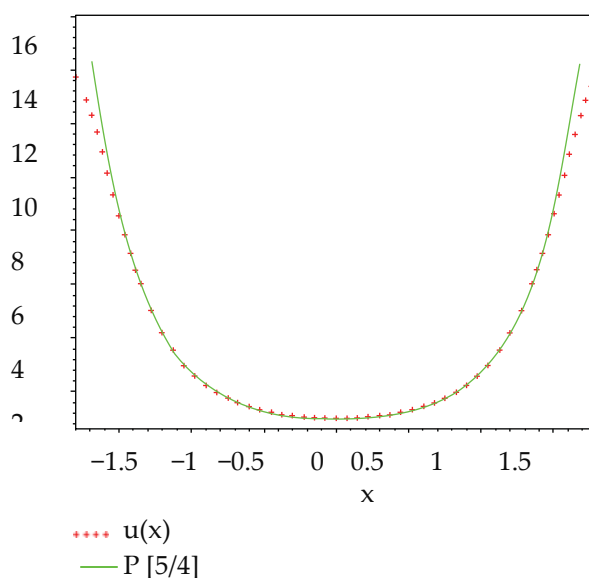
$$xu''(x) + 2u'(x) = 4x^3 u(x) + 6xu(x), \quad 0 \leq x \leq 1. \quad (4.2)$$

By using the fundamental operations of one-dimensional differential transform method in Table 1, we obtained following recurrence relation: [22]

$$\delta(k-1) \otimes (k+1)(k+2)U(k+2) + 2(k+1)U(k+1) \\ = 4\delta(k-3) \otimes U(k) + 6\delta(k-1) \otimes U(k). \quad (4.3)$$

Table 2. Numerical solution of $u(x)$.

x	$u(x)$	$P [5/4]$	$ u(x) - P [5/4] $
0.1	1.010050167	1.010050167	0
0.2	1.040810774	1.040810774	0
0.3	1.094174234	1.094174275	4.1×10^{-8}
0.4	1.173509974	1.173510700	7.26×10^{-7}
0.5	1.284016927	1.284023668	0.000006741
0.6	1.433275840	1.43317285	0.000041445
0.7	1.632060167	1.632251266	0.000191099
0.8	1.895481174	1.896191187	0.000710013
0.9	2.244559634	2.246777235	0.002217601
1.0	2.708333334	2.714285714	0.005952380

**Figure 1.** Values of $u(x)$ and its $P [5/4]$ Padé' approximant. Then,

$$\sum_{l=0}^k \delta(l-1)(k-l+1)(k-l+2)U(k-l+2) + 2(k+1)U(k+1)$$

$$4 \sum_{l=0}^k \delta(l-3)U(k-l) + 4 \sum_{l=0}^k 6\delta(l-1)U(k-1)$$

Discrete Dynamics in Nature and Society

$$U(0) = 1, \quad U(1) = 0, \quad U(2) = 1, \quad U(3) = 0, \quad U(4) = \frac{1}{2}, \quad U(5) = 0,$$

$$U(6) = \frac{1}{6}, \quad U(7) = 0, \quad U(8) = \frac{1}{24}, \quad U(9) = 0, \dots,$$

$$u^*(x) = U(0) + xU(1) + x^2U(2) + x^3U(3) + \dots,$$

$$u^*(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 \dots = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \dots + \frac{1}{n!}x^{2n} \quad (4.4)$$

Power series $u^*(x)$ can be transformed into Pade' series

$$p \left[\frac{4}{5} \right] = \frac{\left(1 + \left(\frac{1}{2}\right)x^2 + \left(\frac{1}{12}\right)x^4\right)}{\left(1 + \left(\frac{1}{2}\right)x^2 + \left(\frac{1}{12}\right)x^4\right)} \\ = \frac{(1 + 0.5x^2 + 0.8333333333x^4)}{(1 + 0.5x^2 + 0.8333333333x^4)} \quad (4.5)$$

Example 4.2 (see Table 3 and Figure 2). Consider the Lane-Emden-type equation

$$y'' + \frac{2}{x}y' + (+8e^y + 4e^{y/2}) = 0, \quad (4.6)$$

with initial values

$$y(0) = 0, \quad y'(0) = 0, \quad (4.7)$$

where the exact solution is $y(x) = -2 \ln(1 + x^2)$.

Multiplying both sides of equation by x ,

$$xy'' + 2y' + 8xe^y + 4xe^{y/2} = 0. \quad (4.8)$$

Also, e^y and $e^{y/2}$ are defined as follows:

$$e^y = 1 + y, \quad e^{y/2} = 1 + \frac{y}{2}. \quad (4.9)$$

Then,

$$xy'' + 2y' + 12x + 10xy = 0. \quad (4.10)$$

By using the fundamental operations of one-dimensional differential transform method in Table 1, we obtain the following recurrence relation[22]:

$$\delta(k-1) \otimes (k+1)(k+2)Y(k+2) + 2(k+1)Y(k+1) + 12\delta(k-1) + 10\delta(k-1) \\ \otimes Y(k) = 0. \quad (4.11)$$

Discrete Dynamics in Nature and Society

Table 3. Numerical solution of $y(x)$.

x	$y(x)$	$P [5/4]$	$ y(x) - P [5/4] $
0.1	-0.01990023712	-0.01990023713	1×10^{-11}
0.2	-0.07841498810	-0.07841499080	2.70×10^{-9}
0.3	-0.1720671642	-0.1720673162	1.520×10^{-7}
0.4	-0.2953112381	-0.2953138699	0.0000026318
0.5	-0.4408387684	-0.4408625091	0.0000237407
0.6	-0.5998683214	-0.6000097952	0.0001414738
0.7	-0.7622819782	-0.7629141475	0.0006321693
0.8	-0.9164312381	-0.9187160286	0.0022847905
0.9	-1.048395758	-1.055408536	0.007012778

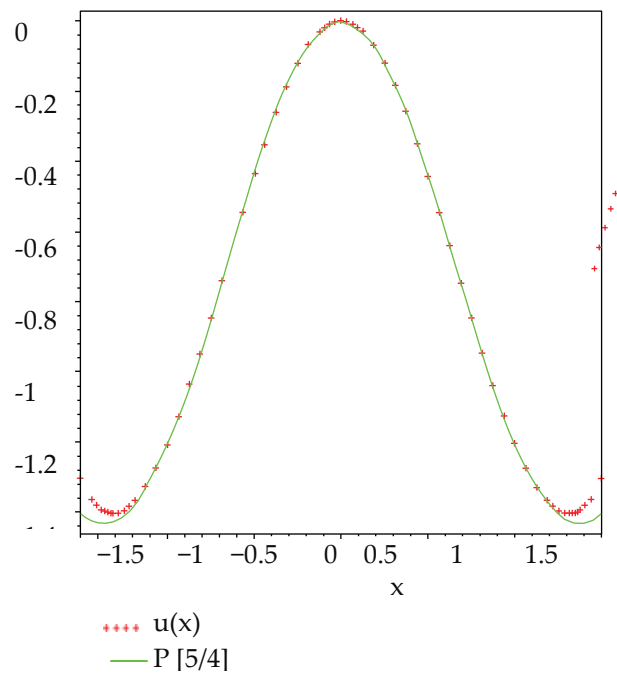


Figure 2. Values of $y(x)$ and its $P[5/4]$ Padé' approximant.

Then,

$$\sum_{l=0}^k \delta(l-1)(k-l+1)(k-l+2)Y(k-l+2) + 2(k+1)Y(k+1) + 12\delta(k-1) \\ + 10 \sum_{l=0}^k \delta(l-1)y(k-l) = 0$$

$$y(0) = 0, \quad y(1) = 0, \quad y(2) = -2, \quad y(3) = 0, \quad y(4) = 1, \quad y(5) = 0,$$

Discrete Dynamics in Nature and Society

$$y(6) = -\frac{5}{21}, \quad y(7) = 0, \quad y(8) = \frac{25}{756}, \dots,$$

$$y^*(x) = y(0) + xy(1) + x^2y(2) + x^3y(3) + \dots,$$

$$y^*(x) = -2x^2 + x^4 - \frac{5}{21}x^6 + \frac{25}{756}x^8 + \dots \quad (4.12)$$

Power series $y^*(x)$ can be transformed into Padé' series

$$p\left[\frac{5}{4}\right] = \frac{((589/704)x^4 - 2x^4)}{(1 + (115/1408)x^2 + (4625/59136)x^4)} \\ = \frac{(0.8366477273x^4 - 2x^4)}{(1 + 0.08167613636x^2 - 0.07820955087x^4)} \quad (4.13)$$

4. Conclusion

This study suggests using a Padé approximation method as a trustworthy and effective way to solve Lane-Emden type differential equations. These equations are a type of nonlinear, second-order ordinary differential equations that often appear in astrophysics. They are commonly used to model the structure of stars, the heat behavior of round gas clouds, and gas spheres at a constant temperature. Because of their unique behavior at the starting point and their complex nature, it is usually very difficult to find exact solutions for Lane-Emden equations. Traditional math methods can be very slow to compute or have problems getting close to tricky points. The Padé approximation gives a useful and very accurate option. The main idea of the Padé approximation method is to

represent a function as a fraction of two polynomials. This helps to describe complicated functions better and more effectively than regular power series expansions. In this case, the differential transform method is used to change the original Lane-Emden equation into a group of algebraic equations[23].

From there, we get a shortened series solution that estimates the answer near the starting point. This shortened series is then used to create the Padé approximant. Unlike Taylor or Maclaurin series that can become inaccurate after just a few terms, the Padé approximant is better at staying accurate over a larger range and can successfully approximate solutions that have poles or unusual behavior. A big benefit seen in this study is how easy it is to use the Padé approximation. The process uses simple math operations, and the resulting function is very similar to the real solution. This efficiency makes it especially good for Lane-Emden equations because it quickly gets close to the important point. Additionally, the Padé approximant not only describes the behavior close to the starting point but also allows for more accurate results at larger values of the independent variable[10].

This is an advantage over methods that rely solely on series. The numbers we got from using the suggested method show clearly that it is accurate and dependable. The calculated results match well with the exact solutions or reference solutions that are already published. This shows that the Padé approximation can give very accurate results using fewer parts than regular methods. It cuts down on computing expenses while keeping accuracy, making it great for real-world uses in physics and engineering. In summary, the Padé approximation method shown in this study is a strong, easy, and useful way to solve Lane-Emden type differential equations. It can deal with difficult problems, work faster, and give good results without using a lot of computer power, making it better than traditional methods. Since it has been successful in this area, the method could be very useful for solving more types of nonlinear differential equations in various scientific fields[24].

REFERENCES

- [1] M. Sajid, T. Hayat, and S. Asghar, 'Comparison between the HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt', *Nonlinear Dyn.*, vol. 50, pp. 27–35, 2007.
- [2] A. Aslanov, 'Approximate solutions of Emden-Fowler type equations', *Int. J. Comput. Math.*, vol. 86, no. 5, pp. 807–826, 2009.
- [3] A. Aslanov, 'Determination of convergence intervals of the series solutions of Emden--Fowler equations using polytropes and isothermal spheres', *Phys. Lett. A*, vol. 372, no. 20, pp. 3555–3561, 2008.
- [4] E. Momoniat and C. Harley, 'Approximate implicit solution of a Lane-Emden equation', *New Astron.*, vol. 11, no. 7, pp. 520–526, 2006.
- [5] S. A. Yousefi, 'Legendre wavelets method for solving differential equations of Lane--Emden type', *Appl. Math. Comput.*, vol. 181, no. 2, pp. 1417–1422, 2006.
- [6] A.-M. Wazwaz, 'A new algorithm for solving differential equations of Lane--Emden type', *Appl. Math. Comput.*, vol. 118, no. 2–3, pp. 287–310, 2001.
- [7] S. Abbasbandy, T. Hayat, A. Alsaedi, and M. M. Rashidi, 'Numerical and analytical solutions for Falkner-Skan flow of MHD Oldroyd-B fluid', *Int. J. Numer. Methods Heat & Fluid Flow*, vol. 24, no. 2, pp. 390–401, 2014.
- [8] M. Dehghan and F. Shakeri, 'Approximate solution of a differential equation arising in astrophysics using the variational iteration method', *New Astron.*, vol. 13, no. 1, pp. 53–59, 2008.
- [9] A. Yıldırım and T. Özıcs, 'Solutions of singular IVPs of Lane--Emden type by the variational iteration method', *Nonlinear Anal. Theory, Methods & Appl.*, vol. 70, no. 6, pp. 2480–2484, 2009.
- [10] J. I. Ramos, 'Series approach to the Lane--Emden equation and comparison with the homotopy perturbation method', *Chaos, Solitons & Fractals*, vol. 38, no. 2, pp. 400–408, 2008.

- [11] S. K. Vanani and A. Aminataei, 'On the numerical solution of differential equations of Lane--Emden type', *Comput. & Math. with Appl.*, vol. 59, no. 8, pp. 2815–2820, 2010.
- [12] A. Yıldırım and T. Özici, 'Solutions of singular IVPs of Lane--Emden type by homotopy perturbation method', *Phys. Lett. A*, vol. 369, no. 1–2, pp. 70–76, 2007.
- [13] J. K. Zhou, 'Differential Transformation and Its Applications for Electrical Circuits', *Huazhong Univ. Press. Wuhan China (In Chinese) google Sch.*, vol. 2, pp. 413–420, 1986.
- [14] G. Adomian, R. Rach, and N. T. Shawagfeh, 'On the analytic solution of the Lane-Emden equation', *Found. Phys. Lett.*, vol. 8, pp. 161–181, 1995.
- [15] R. A. Van Gorder and K. Vajravelu, 'Analytic and numerical solutions to the Lane--Emden equation', *Phys. Lett. A*, vol. 372, no. 39, pp. 6060–6065, 2008.
- [16] B. W. Carroll and D. A. Ostlie, *An introduction to modern astrophysics*. Cambridge University Press, 2017.
- [17] R. Emden, *Gaskugeln: Anwendungen der mechanischen Warmetheorie auf kosmologische und meteorologische Probleme...* BG Teubner, 1907.
- [18] S. Chandrasekhar and S. Chandrasekhar, *An introduction to the study of stellar structure*, vol. 2. Courier Corporation, 1957.
- [19] E. Celik and M. Bayram, 'Arbitrary order numerical method for solving differential-algebraic equation by Padé series', *Appl. Math. Comput.*, vol. 137, no. 1, pp. 57–65, 2003.
- [20] H. J. Lane, 'On the theoretical temperature of the sun, under the hypothesis of a gaseous mass maintaining its volume by its internal heat, and depending on the laws of gases as known to terrestrial experiment', *Am. J. Sci.*, vol. 2, no. 148, pp. 57–74, 1870.
- [21] S. Liao, 'Notes on the homotopy analysis method: some definitions and theorems', *Commun. Nonlinear Sci. Numer. Simul.*, vol. 14, no. 4, pp. 983–997, 2009.
- [22] S. Liao, 'A new analytic algorithm of Lane--Emden type equations', *Appl. Math. Comput.*, vol. 142, no. 1, pp. 1–16, 2003.
- [23] S.-J. Liao, 'The proposed homotopy analysis technique for the solution of nonlinear problems', Ph. D. Thesis, Shanghai Jiao Tong University Shanghai, 1992.
- [24] M. M. Rashidi, S. Abbasbandy, and others, 'Analytic approximate solutions for heat transfer of a micropolar fluid through a porous medium with radiation', *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 4, pp. 1874–1889, 2011.