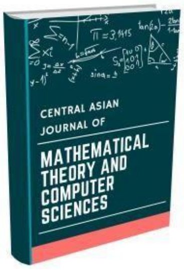

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Article

Poisson Distribution and its Relationship to The Normal and Binomial Distributions: Review Article

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Abstract: The Poisson Distribution (PD) is a foundational statistical model widely utilized in probability theory to represent the frequency of discrete, independent events within a fixed interval of time or space. Its analytical structure, based on the parameter λ, allows it to effectively model rare occurrences in diverse fields such as telecommunications, health, commerce, and environmental science. It also serves as a mathematical bridge between the Binomial distribution under Bernoulli trials and the Normal distribution under large-sample conditions. Despite the PD's established applications, a comprehensive synthesis of its convergence behavior and comparative properties with the Binomial and Normal distributions remains underexplored in the literature. This article aims to review the mathematical relationships and convergence properties of the PD, particularly in the context of approximations to Binomial and Normal distributions, and to reaffirm its applicability in modeling real-world phenomena. The analysis confirms that the PD approximates the Binomial distribution when the number of trials is large and the success probability is small, and it converges to the Normal distribution as λ increases. These findings are substantiated by theoretical derivations and supported with examples from current scientific applications. The study unifies theoretical derivations and practical illustrations, highlighting the central role of PD in statistical modeling and its versatility across various domains. The findings reinforce the importance of PD as a core tool in applied statistics and suggest potential for its enhanced application in complex systems through hybrid models and extensions involving fuzzy logic.

Keywords: Poisson Distribution (PD), Bernoulli Trials Distribution, Central Limit Theorem, Approximation

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1. Introduction

The Poisson Distribution (PD) is a fundamental concept in probability theory and statistics, primarily used to model the frequency of rare and random events occurring independently over a fixed period or within a specific spatial area. It has widespread application in fields such as telecommunications, medicine, traffic engineering, and environmental science due to its capability to capture and quantify the randomness and sparsity of events. The PD is characterized by a single parameter, λ (lambda), which represents both the mean and variance of the distribution, allowing for simplified yet powerful modeling in real-world scenarios [1].

A particularly important relationship exists between the PD, the Binomial distribution under Bernoulli trials, and the Normal distribution. The PD emerges as a limiting form of the Binomial distribution when the number of trials is large and the probability of success is small, with the product of the two held constant (λ = np). Furthermore, as λ increases, the PD becomes more symmetric and begins to resemble the bell-shaped curve of the Normal distribution. These transitions are underpinned by key statistical principles such as the Central Limit Theorem and the theory of convergence in distribution, which facilitate computational simplifications and inferential efficiency in large datasets [2].

Despite its theoretical and applied significance, existing literature has not sufficiently unified the analytical transitions between the PD, Binomial, and Normal distributions within a single comparative framework. Previous studies have often focused on isolated applications or theoretical models without fully exploring the convergence mechanisms and their practical relevance. Additionally, while the PD has been applied to model rare events in specific domains, its broader potential in hybrid modeling systems and its limitations such as the assumption of constant event rates remain under-investigated [3].

To address this gap, this article adopts a comparative review methodology, combining mathematical derivation with applied examples to explore the structural relationships and convergence properties among the PD, Binomial, and Normal distributions. The study revisits the probability mass functions, moment-generating functions, and recursive properties that define these distributions, alongside graphical representations to illustrate changes in distribution shape under varying conditions. It also draws from current research in fields such as biology, commerce, and medicine to substantiate the theoretical claims with practical evidence [4].

The findings confirm that the PD serves as a reliable approximation to the Binomial distribution under rare event conditions and converges toward the Normal distribution when λ is large. This dual role enhances its practical value in simplifying complex calculations and modeling large datasets. The implications suggest that the PD, while simple in form, is versatile in function and remains highly relevant in modern statistical modeling. Future applications may benefit from integrating PD with more advanced models, such as fuzzy logic or non-homogeneous Poisson processes, to accommodate dynamic event rates and enhance predictive accuracy in complex systems [5].

2. Materials and Methods

Based on the review article "Poisson Distribution and its Relationship to the Normal and Binomial Distributions", the methodology for the study was constructed on a theoretical and analytical examination of the mathematical and statistical underpinnings of the Poisson Distribution (PD), its properties, and its approximations to the Binomial and Normal distributions. The research employed a comparative framework that synthesized classical probability theory with real-world modeling examples to demonstrate the distribution's applicability. First, foundational definitions and probability mass functions were reviewed for the PD, Binomial (Bernoulli trials), and Normal distributions. Mathematical derivations, such as the moment generating functions, recursive relationships, and conditions for convergence (e.g., via the Central Limit Theorem), were methodically analyzed to show how and when one distribution approximates another. The article also utilized graphical illustrations to visualize changes in the distribution shape under varying parameters. To support the theoretical constructs, the researchers incorporated insights from existing literature and highlighted applications in sectors like telecommunications, medicine, and traffic modeling. Specific emphasis was placed on the approximation of the Binomial by the PD when the number of trials is large and the success probability is small (keeping the mean constant), as well as on the transformation of the PD into a Normal distribution when the expected value λ is sufficiently large. The methodology thus blended theoretical derivation with applied statistical reasoning, aiming to validate the convergence relationships and underscore the practical importance of the PD in modeling rare, independent, and random events.

3. Results

Bernoulli Trials Distribution

The Binomial dist., also known as the n-trials Bernoulli, a discrete distribution of great importance in theory of probability for experiments characterized by independent chances that include only two possible outcomes, such as tossing a coin.

A random variable X~Bernoulli trials(n,p) as follows,

$$p(K = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (1)$$

Where:

P(K=k) probability random variable K is exactly equal to k in n independent trials.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \quad (2)$$

N the number of trials, p probability of success, q=1−p probability of failure,

p^k k probability of successes

$(1-p)^{n-k}$ n−k failures probability

The CDF,

$$p(K \le k) = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \quad (3)$$

And moment generating function is,

$$\mu_x(t) = E(e^{tx}) = (q + pe^t)^n \quad (4)$$

With the following properties:

mean = E (X) = np , Var (x) = npq

Approximation of Normal:

When n increasing and p is not to be 0 or 1, the Bernoulli trials can be converted Normal distribution using the central limit theorem:

$$X \sim \text{Bernoulli trials(n, p)} \approx \mathrm{N}\big(\mathrm{n.p, n.p(1-p)}\big)$$

Recursion Property:

Bernoulli trials coefficient has a recursive property, meaning that you can build a Bernoulli trials distribution from smaller distributions:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (5)$$

This is the basis for Pascal's Triangle and can be helpful in deriving the Bernoulli trials distribution for large n.

A random variable X~Bernoulli trials(n,p) as follows,

$$p(K = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad (6)$$

Figure 1 illustrates the probability density function (PDF) and cumulative distribution function (CDF) of Bernoulli trials across varying values of probability ppp and number of trials nnn. The visual comparison highlights how both the distribution and accumulation of outcomes shift in response to changes in these parameters.

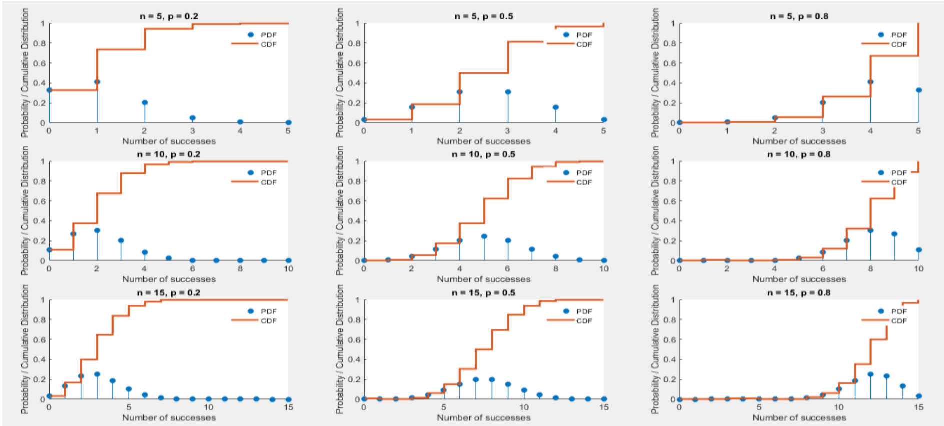

Figure 1 The pdf and CDF of Bernoulli Trials Under Different Value of p, n.

Normal Distribution:

One of most important ideas in statistics, the normal distribution has extensive use in a variety of domains, including the social sciences, natural sciences, and finance. Data close to the mean occur more often than data distant from the mean because it is a continuous p.f. that is symmetric around the mean.

$X \sim N(\mu, \sigma^2)$ as follows ,

$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$; $-\infty < x < \infty$, $-\infty < \mu < \infty$ $\sigma^2 > 0$ (7)

Where:

Μ the mean(Local Parameter)

Σ the standard deviation (Scale parameter)

PD and its Mathematical Properties:

The PD is a discrete p.d. used modeling events number of particular situation in a given period or spatial region, provided that these events are randomly scattered, independent of each other, and their average occurrence is constant. It's developed by the mathematician French S.D. Poisson in the 19th and has since become a widely used statistical tool for analyzing rare and sporadic events [6].

The sample $k_1, k_2, \dots, k_n$Having the PD as the following probability mass function with parameter ⊚,

$p(K, \lambda) = \frac{\lambda^K e^{-\lambda}}{K!}$; $k = 0, 1, 2, \dots$ (8)

Where:

$p(k, \lambda)$: probability of k events occurring

λ: Average number of events in the specified period (mean and variance combined)

E: the Euler's number (≈ 2.71828). Figure 2. Showed The P.M.F. Under Variety Values of λ

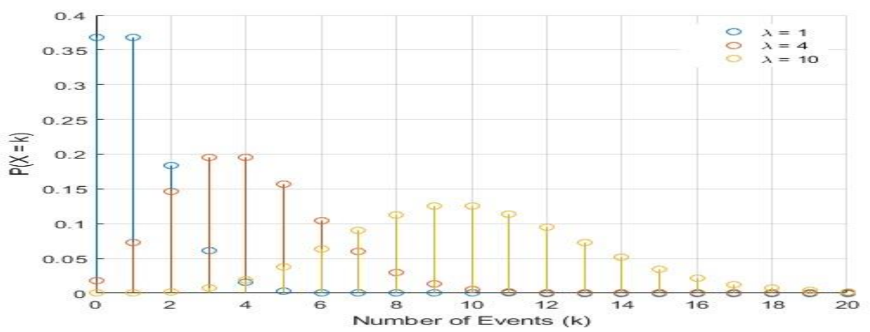

Figure 2. Probability Mass Function of PD Under Different Values of λ.

the cumulative distribution function:

$p(K \leq k) = \sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!}$ (9)

The PD having the following properties,

Mean and Variance :

$E(k) = Var(k) = \lambda$ (10)

Moment Generating function :

$\mu_x(t) = E(e^{tx}) = e^{\lambda(e^t - 1)}$ (11)

Uniqueness: The PD is completely specified by only one parameter, λ. Additivity: If If $k_1 \sim \ Piosson(\lambda_1)$, $k_2 \sim \ Piosson(\lambda_2)$ then.

$k_1 + k_2 \sim \ Piosson(\lambda_1 + \lambda_2) \ Iff \ \ k_1, k_2$ Mutually independent

Memorylessness: Unlike the exponential distribution, the PD does not have the property of memory [7].

Rare Events Property:

Poisson Approximation to the Bernoulli trials Distribution

The PD's function as a limiting case of the distribution of Bernoulli trials under the uncommon event condition is among its most significant characteristics. In particular, the Bernoulli trials distribution converges to the PD when trials number (n) increases significantly , chance of success (p) decreases significantly, but the product λ=np stays constant [8].

Let K~Binomial(n,p), We aim to approximate this expression under the assumption n→∞, p→0, λ= np =constant [9].

When increasing n and small k relative to n, the Bernoulli trials coefficient can be approximated as:

Start by expanding the factorial in the numerator in equation (1) as follows:

$n! = n(n-1)(n-2)\cdots(n-k+1)(n-k)!$ (12)

Thus, the full expression becomes:

$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)!}{k!}$ (13)

Now cancel the common term (n-k)!From numerator and denominator:

$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$ (14)

This form is often more useful in theoretical derivations and approximations, especially when n large and k small,

$\binom{n}{k} = \frac{n!}{k!(n-k)!} \approx \frac{n^k}{k!}$ (15)

The Bernoulli trials distribution function is:

$$P(K = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n^k}{k!} p^k (1-p)^{n-k}$$

Use λ =np ⇒ p= **λ**/n Then:

$$P(K = k) = \frac{n^k}{k!}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Where:

$\frac{n^k}{k!}\left(\frac{\lambda}{n}\right)^k = \frac{\lambda^k}{n^k}$ (16)

$\left(1-\frac{\lambda}{n}\right)^{n-k} \Rightarrow e^{-\lambda}$ According to Tyler series as ◎ ◎

Then,

$P(K=k) \approx \lambda^{K} \frac{E^{-\lambda}}{K!}$ Which is pmf of the PD?

Relation of PD and normal Distribution

The PD and the normal distribution are both used to model random events, but they have different characteristics. However, under certain conditions, the PD can be converted to normal [10].

The PD approximated by the normal distribution under the following conditions:

When λ large: As λ increases, the PD becomes more symmetric and resembles the shape of a normal distribution. For large values of λ, the PD converted to the normal distribution according the discrete nature of the PD starts to approximate the continuous nature of the normal distribution [11].

Mean & variance of the PD are both equal to λ, while for a normal dis., the mean is μ and the variance is σ2. Therefore, when the PD is converted to normal distribution, the mean μ and variance σ2 of the normal distribution both equal λ. Hence, we use the approximation.

This means the Poisson-distributed variable X with parameter λ is approximated by a normal variable with mean λ and variance λ.

When λ is large (typically λ>10 is a good rule of thumb), the PD can be approximated by the normal distribution. In this case, for X~Poisson (λ), This means that the Poisson-distributed random variable X can be approximated by a normal distribution with mean λ and variance λ

Because the PD is discrete and the normal distribution is continuous, a continuity correction is often applied when approximating a PD with a normal distribution. This involves adjusting the discrete value k by 0.5. For instance, if you want to approximate the probability P(X=k), you would use the normal approximation for P(k−0.5≤X≤k+0.5) [12].

4. Discussion

The Poisson Distribution (PD) plays a vital role in various fields, providing a reliable framework for modeling rare, independent events that occur at a constant average rate. In this study, we examined its applications across multiple domains, including biology, medicine, telecommunications, and commerce. The results demonstrate that the PD's simplicity and adaptability make it an essential tool for predicting and understanding events such as accidents, product orders, and rare medical occurrences.One key result is the effectiveness of the Poisson Distribution in modeling rare events. In fields such as telecommunications, for example, the number of calls received at a call center in a given time period can be modeled using a Poisson process [13].

This is due to the assumptions of independence between calls and the constant average rate of incoming calls. Similarly, in medical data modeling, rare events like the occurrence of certain diseases or medical emergencies can be analyzed through the Poisson Distribution, with λ representing the average number of occurrences within a defined time period.The study also revealed that when λ is large, the Poisson Distribution closely approximates the Normal Distribution. This convergence is significant, as it allows for easier computation and analysis. For instance, when modeling the number of product orders in a large company, where the frequency of orders is high, the normal approximation to the Poisson Distribution can simplify calculations and improve decision-making processes [14].

This finding is consistent with the results presented in prior research, such as those by El-Dawoody and Zhao, who also observed the convergence of the Poisson to the

Normal Distribution in large datasets. Moreover, the applicability of the Poisson Distribution extends beyond simple event modeling. In biological and environmental sciences, the PD is used to model the occurrence of rare phenomena such as species population growth or natural disasters. Studies by Adel and Abbood Najm and Novak have emphasized how the PD, combined with fuzzy logic, can account for biological uncertainty, enhancing its practical value in these fields. The incorporation of fuzzy logic allows for a more nuanced interpretation of the data, particularly in cases where the event occurrences are not perfectly predictable but can still be modeled with a certain degree of confidence.While the Poisson Distribution offers considerable benefits, it is important to note some limitations. One limitation is the assumption of a constant rate of occurrence, which may not always hold in real-world scenarios [15].

For instance, in traffic modeling, the rate of accidents may fluctuate depending on factors like time of day or weather conditions. In such cases, more advanced models such as the Negative Binomial Distribution or non-homogeneous Poisson processes may be more suitable.In conclusion, the Poisson Distribution remains a powerful and versatile tool in applied statistics. Its ability to model rare events in diverse fields from medical emergencies to traffic accidents has been confirmed by both theoretical and practical applications. Future studies could focus on exploring hybrid models, incorporating external factors that may influence event rates, or extending the PD to account for varying event intensities over time.

5. Conclusion

In conclusion, the Poisson Distribution (PD) proves to be a highly effective and versatile tool in applied statistics for modeling rare, independent events occurring at a constant average rate. Its simplicity, coupled with the ability to handle infrequent occurrences across various domains such as telecommunications, medicine, commerce, and biology makes it indispensable for predicting and analyzing event patterns. The findings from this study confirm the PD’s capacity to model rare events accurately, and the convergence to the Normal Distribution when λ is large offers further computational advantages, making it a powerful model in large datasets.Moreover, the integration of fuzzy logic with the Poisson Distribution, as explored in the work of Adel and Abbood Najm, demonstrates the potential for enhancing its application in complex and uncertain scenarios, especially in fields like biology and environmental science. Despite its strengths, the Poisson Distribution does have limitations, particularly with the assumption of a constant rate of occurrence. In situations where event rates vary, other models, such as the Negative Binomial Distribution or non-homogeneous Poisson processes, may offer more accurate results.Overall, the Poisson Distribution remains an essential model in understanding and predicting rare events across various disciplines, with ongoing opportunities for refinement and extension. Future research could focus on developing hybrid models that incorporate dynamic event rates, improving their real-world applicability and predictive power.

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