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Article

S-extending Fuzzy Modules

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Abstract: This paper introduces the S-extending fuzzy (fz) modules, i.e. the generalizations of classical semi extending modules in the setting of fuzzy modules. While much recent research on extending and FI-extending fz-modules has been productive, much less is known about how the new categories of CS-fz-modules, CLS-fz-modules, and FI-extending fz-modules are related to fz-modules. In this regard, we use a theoretical and proof based method to prove some necessary properties of S extending fz modules and relationships between these fuzzy module types. Our results show that each S-extending fz module verifies some conditions of direct summand decomposition, and thus it is worth consideration in the theory of modules. The results lay the foundation for further generalizations in algebraic structures, and do so with the possibility of applications in fuz set theory and computational mathematics.

Keywords: Essential Fz-submodules, S-closed Fz-submodules, S-extending Fz-modules

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1. Introduction

Hasan [1], studied extending fz-modules a fz-module χ as (ordinary), generalization of extending module. χ is named extending fz-module if for each closed fz-submodle of χ is a direct summand [1], where a fz-submodule κ of fz-module χ is named closed (shortly $\kappa \leq_c \chi$), if κ has no proper essential fz-submodule, that is $\kappa \leq_e \rho \leq \chi$, then $\kappa = \rho$ [2]. A fz-submodule κ is called essential (briefly $\kappa \leq_e \chi$), if $\kappa \cap \rho \neq 0_1$, for any non-trivial fz-submodule ρ of χ. As a generalization of these concept, we introduce the notion S-extending fz-module. The main goal of this research is to study S-extending fz-modules. This research consists three sections, in the first section contains some definition and properties of fz-sets and fz-modules, and in the second section we give several properties of this class. Section three we study relation S-extending fz-module with the some types fz-modules.

Next throughout this paper, (shortly fuzzy set, fuzzy submodule and fuzzy module is fz-set, fz-submodule and fz-module).

2. Materials and Methods

The research describes a fuzzy (fz) module theory encompassing S-extending fuzzy modules and two approaches to achieving the properties of fuzzy (fz) modules are explored from a theoretical and proof based methodology. The frameworks are then defined, to wit cardinality, closure, ideal, and a singleton; in particular fuzzy sets, submodules and modules are defined in the mathematical foundations of the study. The study systematically reviews the key definitions and existing results from previous studies on fz modules, including extending, CS, FI extending and CLS fz modules, and fills the gaps in current knowledge. To derive new properties and characterizations of S-extending fz-modules, the methodology is based on formal mathematical proofs. The method comprises of building up the logical arguments with founded theorems by validating and maintaining results consistent. Several propositions and theorems pertaining to the structural properties of S-extending fz-modules, their decomposition behavior and other classes of fz-modules are given. Finally, the research also further discusses the S-extending fz-modules in the module theory and examines how they are involved in the fuzzy algebraic structures. Where necessary counterexamples are employed to show limitations of some properties while reinforcing robustness of theoretical framework. The study also treats equivalent assumptions under which S-extending fz-modules coincide with the other well known type of modules. Through such a rigorous proof based approach, S-extending fz-modules are now characterised in a clear way and the research further deepens our knowledge of this algebraic property. This results in bases for further studies on extending fuzzy structures and their application to computational mathematics, fuzzy logic and algebraic systems.

3. Results and Discussion

1. Preliminaries

This section, contains some definitions and properties of fz-sets, fz-modules and fz-submodules, which will used in the next sections.

Definition 1. [3]. Let S be a non-empty set and let I be the closed interval [0,1] of the real line (real numbers). An fz-set χ in S (a fz-subset χ of S) is a function from S into I.

Definition 1.2 [4]. Let x_ℓ : S $\longrightarrow$ I be a fz-set in S, $x \in$ S, $\ell \in [0,1]$, defined by:

$$x_\ell = \begin{cases} 1 & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y \end{cases} \qquad \forall \ y \in \mathrm{S}$$

Then x_ℓ is named a fz-singleton.

If $x = 0$ and $\ell = 1$ then :

$$0_1(y) = \begin{cases} 1 & \text{if} \quad y = 0 \\ 0 & \text{if} \quad y \neq 0 \end{cases}$$

Definition 1.3 [4]. Let κ, ϱ be fz-sets in S, then :

1. $\kappa = \varrho$ iff $\kappa(x) = \varrho(x)$, $\forall x \in$ S.
2. $\kappa \subseteq \varrho$ iff $\kappa(x) \leq \varrho(x)$, $\forall x \in S$
3. $x_\ell \subseteq \nu$ iff $x_\ell(\mathrm{y}) \leq \kappa(\mathrm{y})$, $\forall$ y $\in$ S and if $\ell > 0$, then $\kappa(x) \geq \ell$. Thus $x_\ell \subseteq \kappa$ $(x \in \kappa_\ell)$, (that is $x \in \kappa_\ell$ iff $x_\ell \subseteq \kappa$).

Definition 1.4 [4]. Let κ, ϱ be fz-sets in S, then:

1. $(\kappa \cup \rho)(x) = \max\{(\kappa(\mathrm{x}), \rho(x)\}$, $\forall x \in$ S.
2. $(\kappa \cap \rho)(x) = \min\{(\kappa(x), \rho(x)\}$, $\forall x \in$ S.

$\kappa \cup \rho$ and $\kappa \cap \rho$ are fz-sets in S.

In general if $\{\nu_\alpha, \alpha \in \Lambda\}$, is κ family of fz-sets in S, then:

$$\left(\bigcap_{\alpha \in \Lambda} \kappa_\alpha\right)(x) = \inf\ \{\kappa_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in \mathrm{S}.$$

$$\left(\bigcup_{\alpha \in \Lambda} \kappa_\alpha\right)(x) = \sup\ \{\kappa_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in \mathrm{S}.$$

Definition 1.5 [5]. Let κ be a fz-set in S, $\forall$ t $\in$ [0,1], the set $\kappa_{\mathrm{t}} = \{ x \in S, \kappa(x) \geq \mathrm{t}\}$ is named a level sub-set of κ.

Remark 1.6 [3]. Assume κ, ρ are fz-subsets of a set S, then:

1. $(\kappa \cap \rho)_t = \kappa_t \cap \rho_t$ for any t $\in$ [0,1].
2. $(\kappa \cup \rho)_t = \kappa_t \cup \rho_t$ for any t $\in$ [0,1].

3. $\kappa = \rho$ iff $\kappa_t = \rho_t$, $\forall$ t $\in$ [0,1].

Definition 1.7 [6]. Let f : M $\longrightarrow$ N. Let κ be a fz-set in M, the image of κ denoted by $f(\kappa)$ is the fz-set in N defined by:

$$f(\kappa)(y) = \begin{cases} sup\{\kappa(z) \mid z \in f^{-1}(y)\}\ if \quad f^{-1}(y) \neq \emptyset,\ for\ each\ y \in N \\ 0 \qquad\qquad\qquad\qquad o.w \end{cases}$$

where $f^{-1}(y) = \{x : f(x) = y\}$

and let ρ be a fz-set in N, then the inverse image of ρ, denoted by $f^{-1}(\rho)$ is the fz-set in M defined by : $f^{-1}(\rho)(x) = \rho(f(x))$, $\forall\ x \in$ M.

Definition 1.8 [7]. Let f be a function from a set M into a set N. A fz-subset ν of M is called f-invariant if κ (x) = κ (y), whenever f (x) = f (y), where x, y ∈ M.

Definition 1.9 [4]. A fz-set X of an $\mathcal{R} -$ module M is named fz-module ($\mathcal{R}$-fzmodule M) if :

1. X (x−y) ≥ min {X(x), X(y)}, ∀ x, y ∈. M
2. X (rχ) ≥ X (χ), ∀ χ ∈ M and r ∈ $\mathcal{R}$.
3. X (0) = 1.

Definition 1.10 [5]. Let χ, ω be two an $\mathcal{R}$-fzmodule M (fz-module of an R-module M). ω is called a fz-submodule of χ if $\omega \subseteq \chi$.

Definition 1.11 [4]. If κ is fz-submodule of an $\mathcal{R}$-module M, then the submodule κ_t of M is called level submodule of M, where t ∈ [0,1].

Definition 1.12 [8]. Let $\boldsymbol{\chi}$, γ be R-fzmodules M_1, M_2 individually, define $\boldsymbol{\chi} \oplus \gamma$: $M_1 \oplus M_2 \longrightarrow [0,1]$ $(\chi \oplus \gamma)(h, g)$ = min { $\chi(h), \gamma(g)$, ∀ $(h, g) \in M_1 \oplus M_2$}.

$\chi \oplus \gamma$ is named fz-external direct sum of $\boldsymbol{\chi}$ and γ .

Definition 1.13 [9]. Let χ be $\mathcal{R}$-fzmodule M is named semisimple if, χ is a sum of simple fz-submodules of χ, where χ $\mathcal{R}$-fzmodule M is named simple fz-module if $\boldsymbol{\chi}$ has only one proper fz-submodule, which is 0_1.

Definition 1.14 [8]. A fz-module χ is named uniform fz-module if $\kappa \cap \rho \neq 0_1$, for any non-trivial fz-submodule κ and ϱ of χ.

Definition 1.15 [10]. Let χ be $\mathcal{R}$-fzmodle M, χ is named semiuniform (in short S-uniform) fz-module, if every non-trivial fz-submodule κ of χ is a semiessential fz-submodule of χ.

Definition 1.16 [11]. Let X be a fz-module of an $\mathcal{R}$-module M, if ρ a fz-submodule of X, then ρ is called a semi-essential (in short S-essential) fz-submodule of X, if for all prime fz-submodule η of X and $\rho \cap \eta = 0_1$, then $\eta = 0_1$".

Definition 1.17 [4]. Suppose that κ and ρ be two fz-submodules of $\mathcal{R}$-fzmodule M. The residual quotient of κ and ρ. We define (κ: ρ) by: (κ: ρ) = {r_t: r_t is a fz-singleton of R such that $r_t\rho \subseteq \kappa$}".

Definition 1.18 [8]. Let $\boldsymbol{\chi}$ be non-empty $\mathcal{R}$-fzmodule M. The fz-annihilator of κ denoted by (F-$ann_R\kappa$) is defined by $\{x_t : x \in R, x_t\kappa \subseteq 0_1\}$ t ∈ [0,1], where κ is a proper fz-submodule of χ.

Definition 1.19 [8]. A fz-module $\boldsymbol{\chi}$ of an $\mathcal{R}$-module M is said to be prime fz-module if F-ann κ = F-annχ, for any non-trivial fz-submodule κ of χ".

Definition 1.20 [12]. A fz-module χ of an $\mathcal{R}$-module M is called fully fz-invariant (fully f-invariant), if for every fz-submodule of χ is an f-invariant".

2. S-extending Fuzzy Modules

In section two, we introduced the notion of a S-extending R-fzmodule M, by extended (ordinary) semiextending module. We also state and prove same basic results a but this concept.

Definition 2.1 [13]. An R-module K is named semiextending module, for each submodule of K is S-essential in direct summand ofK.

First, we fuzzify as the follows the definition :

Definition 2.2. A sound χ $\mathcal{R}$-fzmodule M is named S-extending , if every S-essential fz-submodle in a direct summand of χ.

Remark 2.3. Let χ be $\mathcal{R}$-fzmodule M, if κ is an essential fz-submodule of χ, thence κ is S-essential fz-submodule of χ.

Proposition 2.4. Let χ be $\mathcal{R}$-fzmodule M, if χ_t is semiextending module(∀ t ∈ (0,1)), then χ S-extending.

Proof:

Impose κ be a fz-submodule of χ, thence by Proposition (1.6),

κ_t is a submodule of χ_t. But χ_t is a semiextending modle, so there exists N $\leq^{\oplus} \chi_t$ such that $\kappa_t \leq_{sem}$ N. Define B : M ⟶[0,1], by: $\rho(a) = \begin{cases} t & if\ a\ \in N \\ 0 & otherwise \end{cases}$

Clearly ρ a fz-submodule of χ, ρ_t = N. Since N is a direct summand of χ_t, then N⊕H = χ_t, where H ≤ χ_t. Hence $\chi_t = \rho_t \oplus H$, that is $\rho_t \leq^{\oplus} \chi_t$, so $\rho \leq^{\oplus} \chi$ see [1, Lemma 2.1.21]. On the other hand $\kappa_t \leq_{sem} \rho_t$ = N, therefore $\kappa \leq_{sem} \rho$ See [2, Proposition (3.4)].That is χ is S-extending fz-module.

Remarks and Examples 2.5:

1. M = Z_6 as Z-module and χ : M ⟶[0,1], defined by :

   χ(a) = 1, ∀ a ∈ Z_6.

   It is easy to show that χ is a fz-module and χ_t = Z_6 is a semi-extending module see [13, Remarks and Examples 1.2 (1)], so χ is S-extending fz-module by Proposition (2.4).

2. Every fz-CS module is S-extending fz-module.

   Proof:

   It is obviously by Remark *(2.3).*

3. Whole semisimple fz-module is S-extending fz-module .

   Proof:

   Since respective semisimple fz-module is F-CS module see [1, Remarks and Examples 3.1.5 (2)], thence assist Remark (2), it breakpoint S-extending fz-module.

4. Whole uniform fz-module is S-extending fz-module and indecomposable fz-module, " where fz-module χ of an R-module M is called indecomposable fz-module, if 0_1 and χ are the only a direct summands of χ [12]".

   Proof

   Suppose that χ uniform fz-module, then χ is F-CS module and indecomposable fz-module by [1, Proposition 3.1.7], so assist remark (2) we have uniform fz-module a S-extending fz-module .

5. Clearly that every S-uniform fz-module is S-extending fz-module. Let M be a S-uniform $\mathcal{R}$-module M and χ : M ⟶ [0,1], such that χ(a) = 1, ∀ a ∈ M. Then χ is S-extending z-module.

   Proof:

   Let $0_1 \neq \kappa \leq \chi$. Then $\kappa_t \leq \chi_t$ = M, ∀ t ∈ [0,1], by Proposition (1.6). Hence $\kappa_t \leq_{sem} \chi_t$ and, so $\kappa \leq_{sem} \chi$ [12, Proposition(3.4)]. Thus χ is S-extending fz-module.

6. The converse of (5) is not true in general, for example:

   Example. Let M = Z_{24} as Z-module and let χ : M ⟶ [0,1], defined by :

   $\chi(m)$ = 1, ∀ $m \in Z_{24}$.

   It is easy that χ is a fz-module and χ_t = Z_{24} is a semi-extending module by [13, Remarks and Examples 1.2 (6)], so χ is S-extending fz-module by Proposition (2.3) and χ

is not S-uniform, since ∃! fz-submodule κ : M ⟶ [0,1], defined by: $\kappa(n) = \begin{cases} 1 & if \quad n \in \langle \overline{8} \rangle \\ 0 & otherwise \end{cases}$, is not S-uniform of χ.

7. Every fz-module over a semi-simple fz-ring is S-extending.

Proof :

Assume $\mathcal{R}$ is a semi-simple fz-ring and χ be a fz-module , then R_t is semisimple by [7, Proposition 2.7], and χ_t be a module hence χ_t is a semi-extending module by [13, Remarks and Examples 1.2 (10)], imply χ is S-extending fz-module [see Proposition 2.4].

8. If χ, γ be isomorphic fz-module and χ is S-extending fz-module, then γ is S-extending fz-module.

Proposition 2.6. Impose χ be a fz-module of an $\mathcal{R}$-module M, if χ is S-extending fz-module and indecomposable fz-module, then χ is S-uniform fz-module.

Proof:

Suppose $0_1 \neq \kappa \leq \chi$, since χ is S-extending fz-module, then $\kappa \leq_{sem} \rho$, where $\rho \leq^{\oplus} \chi$. But μ is an indecomposable fz-module, then either $\rho = 0_1$ or $\rho = \chi$. But $\kappa \neq 0_1$, therefore $\rho = \chi$. Thus χ is S-uniform fz-module.

We modify definition in [14] as follows :

Definition 2.7. A fz-module χ is named fully essential fz-module if every a non-trivial S-essential fz-submodle of χ is an essential in χ.

Corollary2.8. Assume that χ be $\mathcal{R}$-fzmodule M is S-extending and indecomposable, if χ is fully essential, then χ is uniform fzmodule.

Proof:

It follows easily.

Definition 2.9. [15]. A fz-submodule κ of a fz-module χ is named S-closed fz-submodule (briefly $\kappa \leq_{Sc} \chi$), if has no proper semi-essential fz-submodule of χ.

The following result explains the relationship between S-closed fz-submodule κ with the level κ_t.

Proposition 2.10. [15]. Impose κ fz-submodule of a fz-module χ, if κ_t is S-closed submodule of χ_t, thence κ is S-closed submodule of χ ∀ t ∈ [0,1] .

The following gives theorem a characterizations for S-extending fz-module.

Theorem 2.11. Let χ be $\mathcal{R}$-fzmodule M, then χ is S-extending if for each S-close F-submodule is a direct summand in χ.

Proof:

(⇒) Suppose that χ is S-extending fz-module and $\kappa \leq_{sc} \chi$. Since χ is S-extending fz-module, then there exists a direct summand ρ of χ such that $\kappa \leq_{sem} \rho$. But $\kappa \leq_{sc} \chi$, therefore $\kappa = \rho$.

(⇐) Let κ be any fz-submodule of χ, then $\nu_t$is a submodule of χ_t by Proposition (1.6), implies that there exists an S-closed submodule ρ_t in χ_t see [1, Proposition3.1.3] such that κ_t is a semi-essential submodule of ρ_t, so ρ is an S-closed fz-submodule in χ by Proposition (2.10) and $\kappa \leq_e \rho$. Since $\rho \leq_{sc} \chi$, so ρ is a direct summand in χ (by our assumption).

Proposition 2.12. Let κ and ρ be two a fz-submodules of S-extending fz-module of χ. If $\kappa \cap \rho \leq_{sc} \chi$ then $\kappa \cap \rho$ is a direct summand of κ and ρ.

Proof:

Since χ is S-extending fz-modlue, and $\kappa \cap \rho \leq_{sc} \chi$, then $\kappa \cap \rho \leq^{\oplus} \mu$ imply $\chi = (\kappa \cap \rho) \oplus \kappa$, where $\kappa \leq \chi$. But $\chi = \kappa \cap \rho =$ $[(\kappa \cap \rho) \oplus \kappa] \cap \kappa$, then by [1, Lemma 3.1.4] $\kappa = (\kappa \cap \rho) \oplus (\kappa \cap \kappa)$. Similarly $\rho = (\kappa \cap \rho) \oplus (\kappa \cap \rho)$.

Proposition 2.13. Let χ be a non-empty S-extending fz-modlue and ν , ρ be a fz-submodule such that $\chi = \nu \oplus \rho$ wih ann$\nu_t + ann\rho_t = R$. If χ_t fully essential module, then ν , ρ are S-extending.

Proof:

To show ν is a S-extending fz-module. If ν *and* $\rho = 0_1$, then we have to show that ν is S-extending fz-module. Let $\kappa \leq_{sc} \nu$, if $\kappa = 0_1$, then $\kappa \leq^{\oplus} \nu$. If $\kappa \neq 0_1$, to prove that $\kappa \oplus \rho \leq_{sc} \nu$. Since $\rho \leq_{sc} \rho$ and $\kappa \leq_{sc} \nu$, thence $\kappa \oplus \rho \leq_{sc} \chi$ by [15, Proposition 2.20]. But χ is a S-extending fz-module, therefore $\kappa \oplus \rho \leq^{\oplus} \chi$. Thus there exists β fz-submodule of χ such that $\chi = (\kappa \oplus \rho) \oplus \beta = \kappa \oplus (\beta \oplus \rho)$. That is $\kappa \leq^{\oplus} \chi$, then by [1, Lemma 2.1.21], $\kappa_t \leq^{\oplus} \chi_t$ which implies $\kappa_t \leq^{\oplus} \nu_t$ see [16, Lemma 2.3.4], hence $\kappa \leq^{\oplus} \nu$ by [1, Lemma 2.1.21].

For any finite number of fz-submodule, we can generalize Proposition (2.13).

Proposition 2.14. Impose $\chi = \oplus_{i=1}^{n} \kappa_i$ be non-empty S-extending fz-module, such that κ_i fz-submodule of χ for every i: i= 1, 2, …n

If χ_t is a fully essential module, thence κ_i is S-extending fzmodule as long as ann$\kappa_{t_1} + ann\kappa_{t_2} + \cdots + ann\kappa_{t_n}$= R .

Lemma 2.15. Let χ be a fz-module of an $\mathcal{R}$-module M such that $(F - ann\chi)_t = ann\chi_t$, if χ is a faithful fzmodule, thence χ_t is a faithful module.

Proof:

Since χ is faithful fz-module, then F-annχ = 0_1 and by Remark (1.7) $(F - ann\chi)_t$= $(0_1)_t$, imply that $ann\chi_t$ = 0. Thus χ_t is a faithful module.

Proposition 2.16. Impose χ be $\mathcal{R}$-fzmodule M faithful multiplication and finitely generated. If $\mathcal{R}$is semi-extending ring, breakpoint χ is S-extending fzmodule.

Proof:

Since χ finitely generated fz-module, then by [17, Proposition 2.12], χ_t finitely generated and χ_t multiplication see [9, Proposition 3.3.12], and χ_t faithful for all t ∈ [0,1] by Lemma 2.15, imply that χ_t semi-extending module by [13, Proposition 1,11], hence χ S-extending fz-module by Proposition 2.4.

3. S-extending Fz-modules with The Some Types Fz-modules

This section we study the relationships between S-extending fz-module and CLS fz-module and FI-extending fz-module and we check some condition under which S-extending fz-module and F-CS are equivalent.

S-extending fz-module need not be fz-CS as the following example show.

Example 3.1. Let M = Z-module $Z_8 \oplus Z_2$ and χ : M ⟶ [0,1], such that:

$\chi(n, m)$ = 1, ∀ $(n, m) \in Z_8 \oplus Z_2$.

Let ν : M ⟶ [0,1], define by:

$$\nu(a, d) = \begin{cases} 1 & \text{if } (a, d) \in <(\bar{2}, \bar{1})> \\ 0 & \text{otherwise} \end{cases}$$

It is easy that ν be a F-submodule of μ and $\nu_t = <(\bar{2}, \bar{1})>$ ∀ t ∈ [0,1]. But $<(\bar{2}, \bar{1})>$ is closed submodule of M, which can be checked easily. Hence ν is closed see[1, Remarks] and Examples [17, 2.1.15]. But ν_t is not a direct summand of χ_t = M, so that ν is not direct summand of χ by [1, Lemma 2.1.21]. Thus χ is not fz-CS module and χ is S-extending fz-module, since for all S-closed fz-submodule are:

ν_1, ν_2 and χ, where

ν_1 : M ⟶[0,1], ν_2 : M ⟶[0,1], defined by:

$$\nu_1(a, b) = \begin{cases} 1 & \text{if } (a, b) \in \langle(\bar{0}, \bar{1})\rangle \\ 0 & \text{otherwise} \end{cases}$$

And

$$\nu_2(a, b) = \begin{cases} 1 & \text{if } (a, b) \in \langle(\bar{4}, \bar{1})\rangle \\ 0 & \text{otherwise} \end{cases}$$

All of them are a direct summand.

The following Proposition shows that under certain condition S-extending fz-modules implies fz-CS module.

Proposition 3.2. Impose χ be a fz-module provided that any non-trivial S-essential fz-submodle of χ is a fully essential. Thence χ is fz-CS iff χ is S-extending fz-module.

Proof:

($\Rightarrow$) It is omitted.

($\Leftarrow$) Let $0_1 \neq \nu \leq \chi$, then $\nu \leq_{sem} \rho \leq^{\oplus} \chi$; that is ρ is a non-empty fz-submodule of χ. On the other hand ρ is a fully essential fz-submodule, imply that $\nu \leq_e \rho$, then χ is fz-CS module.

"Recall that an R-module M is called fully prime, if every proper submodule of M is a prime submodule [18]".

Remark 3.3. Let ν be a fz-submodule of a fz-module χ and χ_t be a fully prime module. If $\nu_t \leq_{sem} \rho_t$, then $\nu \leq_e \rho$.

Proof:

Let ν be a fz-submodule of ρ, then by Proposition (1.6) ν_t is a fz-submodule in ρ_t implying ν_t is an essential submodule in ρ_t see [13, Proposition 2.2.7], therefore ν is an essential fz-submodule in ρ by [11, Proposition 2.2].

Proposition 3.4. Let χ be S-extending fz-module of an $\mathcal{R}$-module M and χ_t be a fully prime module, then χ is fz-CS module iff χ is S-extending fz-module.

Proof:

($\Leftarrow$) Let $0_1 \neq \nu \leq \chi$ and since χ is S-extending fz-module, then $\nu \leq_{sem} \rho \leq^{\oplus} \chi$, where $\rho \leq \chi$. But χ_t is a fully prime module, then ν is an essential fz-submodule of ρ by Remark (2.13). Thus χ is fz-CS module.

($\Rightarrow$) It is omitted

Proposition 3.5. Let χ be S-extending fz-module provided that every fz-submodule ν of χ, there exists $\rho \leq_{sc} \chi$ with $\nu \leq_e \rho$. Then χ is fz-CS module.

Proof:

Let ν be a fz-submodule of χ, then by assumption there exists ρ fz-submodule of χ such that $\rho \leq_{sc} \chi$ and $\nu \leq_e \rho$. But χ is S-extending fz-module, then $\rho \leq^{\oplus} \chi$, hence χ is fz-CS module.

The following proposition show that under certain condition CLS-fz modules implies S- extending fz-modules. Before that we need are following definitions.

Definition 3.6 [1]. Let χ be a fz-module such that : Z(χ) = {$x_t \subseteq \chi$: F-ann(x_t) is an essential fz-ideal of R} is named fz-singular submodule of χ. If Z(χ) = χ and χ is named non-singular if Z(χ) = 0_1.

Definition 3.7 [1]. Let χ be $\mathcal{R}$-fzmodule M, χ is named CLS-fzmodule if for each $\mathcal{Y}$-closed fz-submodule a direct summand in χ, where a fz-submodule ν is named $\mathcal{Y}$-closed fz-submodule, if χ/ν is non-singular fz-module .

Proposition 3.8. Assume χ be a non-singular $\mathcal{R}$-fzmodule M. If χ is a CLS-fz module, thence χ is S-extending fz-module.

Proof:

Suppose χ be a CLS- fz module, $\kappa \leq_c \chi$. Since χ is a non-singular fz-module, then κ is an Y-closed fz-submodule by [1, Proposition 2.5.9]. But χ is a CLS-fz module, thus $\kappa \leq^{\oplus} \chi$.

We modify definition in [19] as follows :

Definition 3.9. Let χ be $\mathcal{R}$-fzmodule M, χ is named fz-duo, if for each fz-submodle of χ is fully fz-invariant.

Definition 3.10. A sound fz-module χ of an $\mathcal{R}$-module M is named FI-extending fz-module, if for each fully invariant is an essential in a direct summand of χ.

Remark 3.11. Every fz-CS module is an FI-extending fz-module.

Proposition 3.12. [20]. Let χ be $\mathcal{R}$-fzmodule M, if χ be a duo and FI-extending , thence χ is S-extending fz-module.

Proof:

Suppose $\kappa \leq \chi$**.** Since χ is duo fz-module, thence κ fully fz-invariant. But χ is an FI-extending fz-module, thus $\kappa \leq_e \rho$, where ρ is a direct summand. Thence χ is S-extending fz-module.

The converse of Proposition 3.12 is true under certain condition as the following Proposition.

Proposition 3.13. Impose χ be a fz-module of an $\mathcal{R}$-module M provided that every non-trivial is a direct summand is a fully essential in χ. If χ is S-extending fzmodule, thence χ is FI-extending.

Proof:

Suppose that χ is S-extending fzmodule, thence by Proposition 3.2 χ is fz-CS module, implying χ is FI-extending fz-module.

4. Conclusion

The main aim of this paper we introduce the notion of S-extending fzmodule. We prove some fundamental result about this concept, which are analogous to the ordinary semiextending modules. Also we give some characterizations of S-extending fz-module (See propositions 2.4 and 2.11), and illustrate the relationship between S-extending fzmodule and aboth of F-SC module with CLS fzmodule and FI-extending (See Remark 2.5, Example 3.1). Some equivalent statement for S-extending under certain condition are given in Proposition 3.2, Proposition 3.4, Proposition 3.8 and Proposition 3.13.

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