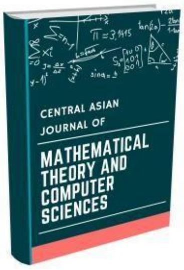

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Article

Survival Function Estimation for Weibull Distribution Based on Granular Hesitant Fuzzy Set

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Abstract: This paper proposes a set of fuzzy with granular hesitation (FSGH) that combines the features of set of fuzzy with Hesitation (FSH) and Fuzzy Granular set (FGS) to allow the analysis of fuzzy data by considering multiple possible values of belonging at different levels of granularity, capturing the details of the data and the ambiguity associated with determining the exact belonging, making it particularly useful in the context of complex decision making. The suggested set is applied to the Weibull distribution, and Monte Carlo simulation studies are conducted to explain behavior of the suggested approach and compare it to the standard sample method. It is found that the suggested FSGH outperforms the typical Weibull distribution data set, with the approach achieving the lowest error criteria, particularly in big data sets. The method's accuracy improves as the frequency and granularity increase.

Keywords: Survival function, Weibull distribution, Set of fuzzy, Hesitant set, Granular hesitation set, Estimation

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1. Introduction

Doubt appears almost everywhere in our daily lives, hence several approaches have been developed to identify, display, change, and deal with doubt. Probability and theory of Set of fuzzy are two of the most often used ways to dealing with uncertainty, since they have been proposed to explain statistical fuzziness and uncertainty. The two models give fundamentally different information. Probability theory communicates relative frequencies, while theory of Set of fuzzy identifies commonalities between items and poorly defined properties. Since its introduction by Zadeh (1965), theory of Set of fuzzy has emerged as one of the most successful decision-making methodologies for coping with uncertainty and ambiguity. Following Zadeh's pioneering work, theory of Set of fuzzy has evolved in a number of ways, the most notable of which is the representation of degrees of membership in the basic Set of fuzzy. Torra (2010) created a Set of fuzzy approach that varies from multi-Set of fuzzy by relying on Set of fuzzy expansions. Liao (2018) used Set of fuzzy intuition to the selection of fire rescue techniques. Liao et al. (2018) introduced a Set of fuzzy intuition for emergency management. Petry (2024) proposed a confidence intuition for Set of fuzzy including geographical and temporal data. Pratama (2024) utilized the circular intuitionistic Set of fuzzy to determine the radius of the unit circle. The literature has several proposals for Set of fuzzy extensions and expansions, such as intuitionistic Set of fuzzy, type 2 Set of fuzzy, and multi-Set of fuzzy. In this paper, we define Set of fuzzy with Hesitations.

2. Intuitionistic Set of fuzzy (IFS)

In traditional Set of fuzzy, each element x in the universal set Z is associated with a single membership score $m_Z(z)$ ranging from 0 to 1, reflecting the extent to which that element belongs to the set. However, in Intuitionistic Set of fuzzy (IFS), this representation is extended by adding another score known as the non-membership score $m_V(z)$, which expresses the extent to which the element does not belong to the set. The intuitive Set of fuzzy A may be represented mathematically as follows: [Torra, 2010, 350] [Petry, 2024, 3]

$A = \{z \in Z : \langle z, m_X(z), m_V(z) \rangle\} \ldots (1)$

Where $0 \leq m_X(z) + m_V(z) \leq 1$, $\forall z \in Z$, , x denotes the same member from the complete set Z. It might be anything, including a number, an item, or an idea, $m_Z(z)$ level of elemental membership z in a fuzzy set Z, which ranging among 0 to 1, $m_V(z)$ level of non-set membership of element z in the set Z, which ranging among 0 to 1. If $m_Z(z) + m_V(z) < 1$, the difference measures the level of hesitation or doubt, and is computed as follows: [Ding et al., 2021, 9075]

$P(z) = 1 - m_Z(z) - m_V(z) \ldots (2)$

$P(z)$ is the amount of uncertainty that cannot be assigned to the degree of membership or non-membership, and it is commonly used in intuitive Set of fuzzy to represent elements in a way that accounts for both membership, non-membership, and hesitation, allowing for a more in-depth analysis of ambiguous or uncertain information.

If we have two fuzzy sets A, B recognized by $m_A(z), V_A(z), m_B(z), V_B(z)$, then the operations on these sets are: [Torra, 2010, 351][Pratama et al., 2024, 12262][Torra & Narukawa, 2009, 20-21]

Union: If A and B are two intuitive set of fuzzy with recognized by membership and non-membership functions, then the union A∪B The computation of the new membership function for element z occurs when sets A and B are joined using the union operation. The membership degree of the element z in the union highest value among the membership degrees in the sets. $m_A(z)$,$m_B(z)$, The non-membership degree of x in the union is the smallest value among the non-membership degrees for sets $V_A(z)$ and $V_B(z)$,

$A \cup B = \{z, Max(m_A(z), m_B(z) , Min(V_A(z), V_B(z)\} \ldots (3)$

Intersection: If A and B are two intuitive Set of fuzzy with unique membership and non-membership functions, the intersection of A and B yields the new membership function of element z. In the intersection, the membership degree of the element x is the lowest value between the membership degrees in the sets $m_A(z)$ and $m_B(z)$, while the non-membership degree is the highest value between the non-membership degrees in the sets $V_A(z)$ and $V_B(z)$.

$A \cap B = \{x, min(m_A(x), m_B(z) , Max(V_A(z), V_B(z)\} \ldots (4)$

Complement: The complement of a set of fuzzy is the exchange of the degrees of membership and non-membership for each element x in the set. If A is a Set of fuzzy with special membership and non-membership functions $m_A(z)$, $m_B(z)$, then the degree of membership Ac becomes the original degree of non-membership $m_{A^c}(z)$, and the degree of non-membership $m_V(z)$ in the complement becomes the original degree of membership $m_B(z)$.

In a set of fuzzy, the complement is the exchange of the degrees of membership and non-membership for each element z in the set. If the set A is a set of fuzzy with a special membership and non-membership function $m_A(z)$and $m_B(z)$, then the degree of membership A^c becomes the original degree of non-membership $m_{A^c}(z)$, and the degree of non-membership $m_V(z)$ in the complement becomes the original degree of membership $m_B(z)$,

$A^c = \{z, (V_V(z), (m_A(z))\} \ldots (5)$

Intersection using the multiplicity operation (Intersection-⊗): describes the dynamic between members and those who choose not to be members. This operation is important in data analysis and fuzzy processing where information from multiple sources is combined and how they overlap or agree with each other is evaluated in a precise and systematic way. If the two sets A and B represent two intuitive Set of fuzzy with special membership and non-membership functions, then the intersection of A⊗B, the new membership function for z is calculated. The membership degree of z in the intersection is the minimum value between the membership degrees in two sets $m_A(z)$, $m_B(z)$, the non-membership degree of the element z in the intersection is the maximum value between the non-membership degrees in the two sets $V_A(z)$ and $V_B(z)$,

$$\mathrm{A} \otimes \mathrm{B} = \{\langle \mathrm{z}, \mathrm{m_A}(\mathrm{z}) \cdot \mathrm{m_B}(\mathrm{z}), \mathrm{V_A}(\mathrm{z}) + \mathrm{V_B}(\mathrm{z}) - \mathrm{V_A}(\mathrm{z}).\mathrm{V_B}(\mathrm{z}) \rangle\}$$

$$= \{\langle \mathrm{z}, \mathrm{m}_{A\otimes\mathrm{B}}(z), \mathrm{V}_{A\otimes\mathrm{B}}(z), \mathrm{V}_{A\otimes\mathrm{B}}(z) \rangle\} \ \dots (6)$$

3. Fuzzy Multi-sets (FMS)

It builds upon the idea of the classic set that aims to represent data in which elements may be repeated with degrees of membership associated with each repetition, providing a flexible tool for dealing with information that includes both uncertainty and repetition. The fuzzy multi-set can be expressed as follows: [Sebastian, Sabu ; T. V. Ramakrishnan, 2010, 2472][Apostolos, 2022, 2]

$A = \{z_i,\ (\mathrm{m}_A(x), \mathrm{i}), z_i \in Z\ \ , \mathrm{m}_A(z), \mathrm{i} \in [0, 1], n_i\ \in N\} \dots$ (7)

Where n_i signifies the frequency at which the component z_i appears in the set Z, N the set of natural numbers.

4. Set of fuzzy with Hesitation (HFS)

Torra (2010) presented a new generalized Set of fuzzy called the set of fuzzy with Hesitation (HFS), which is utilized in decision making in hesitant situations with many options. It is an extension of typical set of fuzzy in which each element in the interval [0, 1] is assigned a collection of alternative affiliation scores rather than a single affiliation score, indicating hesitancy or ambiguity regarding the precise value of affiliation. HFS is employed in cases when specialists or decision makers cannot definitely assign an exact affiliation value; nevertheless, they present a range of probable values to enable more capacity to cope with ambiguous or reluctant evidence. It varies from the multiple set of fuzzy in that it considers a collection of membership scores that indicate uncertainty in the phenomena being studied, while the multiple Set of fuzzy has affiliation values within a certain range and memberships are not exactly defined. Its mathematical model is: [Torra, 2010, 529] [Zheng & Pedrycz, 2022, 61] [Liao et al., 2018, 2]

If we have a collection of k membership functions, we define the Set of fuzzy associated with k as follows:

$$H_k(z) = \cup_{m<k}\{m_1, m_2, .., m_k\} \dots (8)$$

It means the union of all possible degrees of membership given by the functions m in the set k for the element z. If you have several membership functions that give different degrees of membership for the element x, then $\mathrm{H_k}(\mathrm{z})$ will be the set that includes all these possible values, which reflects the hesitation or uncertainty about the true membership of z.

Then $H_k(z)$ represents the set of hesitation of the degree to which z belongs to the set, and m_i are the possible values of membership, with the condition that $m_i \in [0,1]$. The Set of fuzzy is called an empty set if $H_k(z) = \{0\}\ \ \forall z\ in\ Z$ in Z and is called a complete set of fuzzy if $H_k(z) = \{1\}\ \ \forall z\ in\ Z$. It is called a completely unknown $H_k(z) \in [0, 1]\ \ \forall z\ in\ Z$ in case that it is not known how to determine an exact value for membership. It is called a trivial Set of fuzzy if $H_k(z) = \{\emptyset\}\ \ \forall z\ in\ Z$. The degrees of membership for set of fuzzy are determined according to the set of fuzzy rules that are used for the purpose of dealing with ambiguity to help in dealing with cases where the data is inaccurate or when it is difficult to determine the absolute boundaries between different cases, which allows for making more flexible decisions compared to traditional systems that rely on precise rules, as follows:

$$Rule\ R_i : If\ Z\ is\ A_i\ then\ Y\ is\ B_i\ \dots (9)$$

SinceA_i and B_i are two set of fuzzy, each of which has a membership function m_{A_i} and m_{B_i}respectively on the domains Z and Y, then if Z is A_i, it means that the variable Z belongs to the Set of fuzzy A_i to a certain degree determined by the membership function m_{A_i}. And if Y is B_i, it means that the variable Y belongs to the Set of fuzzy A_i to a certain degree determined by the membership function m_{B_i} based on the rule that determines the relationship between X and Y.

Fuzzy rules are the basis of fuzzy systems that allow inputs and outputs to be linked in a way that takes ambiguity and uncertainty into account, making them useful in applications that deal with fuzzy or imprecise data.

If the input variable is z_0, then the fuzzy output based on the fuzzy rule is as follows:

$$\hat{y}(z_0) = \cup_{m_{A_i}(x_0)} \wedge B_i\ \dots (10)$$

Equation (1) means that the system takes the union of all possible outcomes from different rules. In other words, the system takes into account all the rules R_i related to z_0, and then combines or combines these outcomes to get the final result. $\wedge$ intersection or "AND" operation means that the result depends on the minimum (or intersection) between the degree of belonging of x_0 to group A_i and the degree of belonging of output Y to group B_i.By using Set of fuzzy, $\hat{y}(z_0)$ can be defined as the Set of fuzzy associated with the set $\{m_{A_i}(z_0) \wedge B_i\}_i$.

5. Fuzzy Granular Set (FGS)

In Theory of Set of fuzzy, the term "Granular" is used to refer to the idea of "granularity" or "Hierarchy". It expresses how data or knowledge is divided or partitioned into smaller and more precise units (granules), where these units can be overlapping or intertwined, allowing for dealing with vague or ill-defined data or concepts in a more flexible and accurate way. From the perspective of Set of fuzzy with Hesitations or even in the field of artificial intelligence in general, granular analysis can be useful in improving the accuracy of models when dealing with uncertain or incomplete data. [Zhang &. Yang, 2020, 70]

If we have a set Z divided into i set of fuzzy at different granularity levels j as follows:

$$Z = \cup_{i=1}^{n} A_i^j\ \dots (11)$$

Where i used to specify the subset within a given level of granularity, and j is used to specify the level of granularity itself.

6. Proposed Set of fuzzy with granular hesitations (FSGH)

A new set can be proposed that combines the features of Set of fuzzy with Hesitations (HFS) and Fuzzy Granular Sets (FGS) to take advantage of the capabilities of HFS to handle hesitation and uncertainty, and the capabilities of FGS to partition data into finer units across multiple levels of granularity. The FSGH mechanism allows for the analysis of fuzzy data by considering multiple possible values of belonging at different levels of granularity. This method captures both the details of the data and the ambiguity associated with determining the exact belonging, making it particularly useful in the context of complex decision making. The Fuzzy Granular Set (FSGH) is represented as follows:

$$GH_j^k = \cup_{m<k} H_j^k(z) = \cup_{m<k}\{m_1{}^j, m_2{}^j, .., m_k{}^j\}\ \dots (12)$$

Where GH_j^k is a fuzzy granular set of element x at granularity j and hesitation $H_j^k(z)$ is a Fuzzy Granular Set at granularity j ${m_i}^j$, represents the different possible degrees of belonging at granularity j.

Table (1) shows a comparison between the types of Set of fuzzy:

Table (1) Comparison between types of Set of fuzzy

Feature / Type	**Traditional Set of fuzzy**	**multi Set of fuzzy**	**Set of fuzzy with Hesitation**	**Fuzzy Granular Set**	**Set of fuzzy with granular hesitation**
Degree of membership	One value between [0, 1]	Set of fuzzy of values (Type-1) between [0, 1]	A set of possible values between [0, 1]	A single value between [0, 1] within granular units.	A set of possible values within granular units.
Dealing with fuzziness	Simple, deals with uncertainty in the belonging of a single element	Addresses double fuzziness: uncertainty in degree of membership	Deals with hesitation or doubt in determining the degree of membership	Handles fuzziness across multiple levels of precision.	Addresses hesitation and fuzziness across multiple levels of resolution
representation	Simple membership function	Multivalued membership function	List of possible values for the degree of membership	Fuzzy clusters distributed across levels of granularity.	Fuzzy clusters distributed across levels of granularity
Usage	Simple decision making or fuzzy control Simple	Complex applications such as intelligent systems or advanced control.	Surveys, decision making where there is hesitation	Multilevel or domain data analysis	Complex decision making requires analysis of ambiguous and uncertain data across multiple levels
Complexity	Simple	More complex due to the need to define a subset.	Medium complexity, based on hesitation of values	Medium complexity, depends on granularity levels	Highly complex, combining the features of hesitation and granularity

example	Representing "person's height" as "tall" with an membership score of 0.7	The degree of membership itself is 0.7 uncertain and ranges between [0.65, 0.75]	"Person Height" may be "tall" with a score of 0.7, 0.8, or 0.75.	"Person height" is analyzed at multiple levels of granularity	A person's height may be tall or short to varying degrees at multiple levels of granularity
Calculation method	Simple membership function	Nested membership function, requires more complex calculations	Repetition between possible values of the degree of membership	Data is divided into units or groups at levels	Repeating possible values within different granular units

7. Weibull Distribution:

One of the most common probability distributions used in modeling data related to survival or failure time in many applications such as reliability engineering, medical survival studies, and analysis of data related to failures. This distribution was derived by the Swedish scientist Waloddi Weibull in 1939 AD. [Kersey: 2010 ,7][[Kundu & Howlader: 2010, 1548]

The r.v y has a Weibull distribution with the following probability density function:

$$g(y, \alpha, \lambda) = \alpha\lambda y^{\alpha-1} \exp(-\lambda y^{\alpha}) \quad ; \; y > 0 \qquad \text{... (13)}$$

α>0 is the Shape Parameter and λ>0 is the Scale Parameter. y the random variable survival time.

The cumulative distribution function is:

$$F(y, \alpha, \lambda) = 1 - \exp(-\lambda y^{\alpha}) \,; \; y \geq 0 \qquad \text{... (14)}$$

The survival function:

$$S(y, \alpha, \lambda) = 1 - F(y, \alpha, \lambda) = \exp(-\lambda y^{\alpha}) \,; \; y \geq 0 \qquad \text{... (15)}$$

The hazard function:

$$h(y, \alpha, \lambda) = \alpha\lambda y^{\alpha-1} \,; \; y \geq 0 \qquad \text{... (16)}$$

The moment of order r about the origin is:

$$EX^{r} = \int_0^{\infty} X^{r} f(x)\,dx = \frac{r\Gamma\left(\frac{r}{\alpha}\right)}{\alpha\lambda^{\frac{r}{\alpha}}} \quad ; \quad r=1,2,3,\ldots \qquad \text{...(17)}$$

Then,

$$EX= \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\lambda^{\frac{1}{\alpha}}} \qquad \ldots(18)$$

$$EX^2 = \frac{2\Gamma\left(\frac{2}{\alpha}\right)}{\alpha\lambda^{\frac{2}{\alpha}}} \qquad \ldots(19)$$

$$Var(X) = \frac{r\Gamma\left(\frac{r}{\alpha}\right)}{\alpha\lambda^{\frac{r}{\alpha}}} - \left(\frac{\Gamma^2\left(\frac{1}{\alpha}\right)}{(\alpha\lambda^{\frac{1}{\alpha}})^2}\right)] \qquad \ldots(20)$$

8. Maximum Likelihood Estimation:

Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$be a Set of fuzzy with granular hesitation (FSGH) having Weibull distribution with the following probability density function in equation (13)

The likelihood function L(θ) for observations as following,

$$L(\tilde{\theta}) = \prod_{i=1}^{\tilde{n}} g(\tilde{y}_i, \alpha, \lambda)$$

$$= \prod_{i=1}^{\tilde{n}} \alpha\lambda \tilde{y}_i^{\,\alpha-1} \exp(-\lambda \tilde{y}_i^{\,\alpha})$$

$$= (\alpha\lambda)^{\tilde{n}} \prod_{i=1}^{\tilde{n}} \tilde{y}_i^{\,\alpha-1} \exp(-\lambda \tilde{y}_i^{\,\alpha}) \qquad \ldots(21)$$

Then the Log $L(\alpha, \lambda)$ is,

$$Log\big(L(\alpha,\lambda)\big) = \tilde{n}\log(\alpha) + \tilde{n}\log(\lambda) + (\alpha - 1)\sum_{i=1}^{\tilde{n}} \log(\tilde{y}_i) - \lambda\tilde{y}_i^{\ \alpha} \quad \dots \ (22)$$

By derivative the equation (14) according to parameter α, λ and equal to zero we obtain ,

$$\frac{\partial Log\big(L(\widehat{\alpha,\lambda})\big)}{\partial\hat{\alpha}} = \frac{\tilde{n}}{\hat{\alpha}} - \sum_{i=1}^{\tilde{n}} \log(\tilde{y}_i) - \lambda\tilde{y}_i \log(\tilde{y}_i^{\ \alpha}) = 0 \ \dots (23)$$

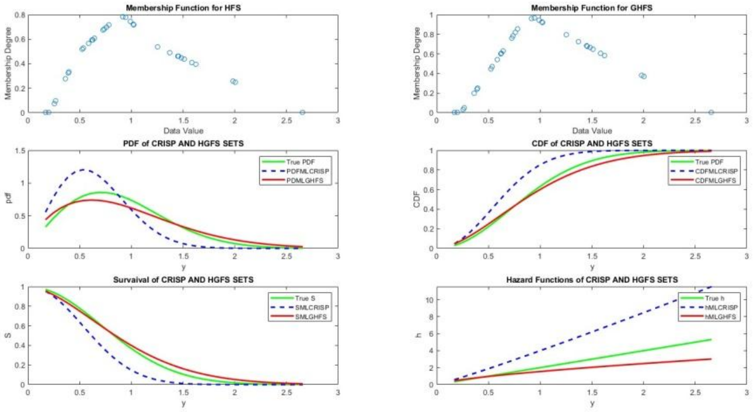
$$\frac{\partial Log\big(L(\widehat{\alpha,\lambda})\big)}{\partial\hat{\lambda}} == \frac{\tilde{n}}{\hat{\lambda}} - \tilde{y}_i^{\ \alpha} = 0 \dots (24)$$

5. Simulation experiments:

Monte-Carlo simulation was used to test the proposed method on Weibull distribution. First, the default values of Weibull distribution, which are the scaling parameter (α) and the shape parameter (λ), were determined. Then, sample sizes (n=35, 50, 75, 100) were randomly selected from Weibull distribution using the inverse of the distribution function. Next, we determine the frequency levels and the number of iterations for each frequency level using fixed values. Then, we create a frequency Set of fuzzy (HFS) for each frequency level by selecting a random subset of the data and iterating each element according to the specified frequency level. Then, we calculate the granular frequency Set of fuzzy (FSGH) by merging the Set of fuzzy across the frequency levels. Then, the belonging values for each element of the data were calculated using the triangular belonging function for both HFS and FSGH. Then, we calculate the modified vector based on the integration of the original probability density function. Finally, we use the maximum likelihood (MLE) method to estimate the parameters of the Weibull distribution for both the original and modified vectors. We then display the estimated values and generate and plot the PDF and CDF of the Weibull distribution for both the original and modified vectors, calculating the MSE and MAPE for each. As follows:

Table 2: Estimated parameters, MSE, MAPE for PDF, CDF, S and h of Weibull distribution under FSGH set and Crisp set for α = 1 and λ = 2 for all sample sizes and j=3, i=2

j	i	n	Criteria Data set	Est. Para.		MSE (PDF)	MAPE (CDF)	MSE (CDF)	MAPE (CDF)	MSE (S)	MAPE (S)	MSE (h)	MAPE (h)
				$\hat{\alpha}$	$\hat{\lambda}$								
3	2	35	FSGH	**0.17**	**2.47**	**0.0083**	**7.6794**	**0.2353**	**7.4118**	**0.0916**	**10.6565**	**0.0016**	**9.4798**
			Crisp	**0.14**	**2.92**	**0.0391**	**8.9859**	**0.4391**	**9.9707**	**12.1847**	**11.9491**	**2.9385**	**10.9270**
		50	FSGH	**1.10**	**2.02**	**0.0038**	**2.2380**	**0.0017**	**3.1867**	**0.0017**	**1.1587**	**0.0013**	**2.0005**
			Crisp	**1.23**	**2.05**	**0.0027**	**3.3616**	**0.0295**	**7.9295**	**0.0295**	**2.8593**	**2.3316**	**6.6493**
		75	FSGH	**1.09**	**2.04**	**0.0009**	**2.1719**	**0.0011**	**2.1133**	**0.0029**	**1.0687**	**0.0012**	**1.4174**
			Crisp	**1.41**	**2.37**	**0.0025**	**3.2064**	**0.0227**	**5.3414**	**0.0227**	**1.5585**	**2.2626**	**5.3448**
		100	FSGH	**1.02**	**2.11**	**0.0008**	**2.0795**	**0.0002**	**1.7896**	**0.0002**	**1.0040**	**0.0011**	**1.2772**
			Crisp	**1.33**	**2.19**	**0.0024**	**2.1119**	**0.0410**	**4.3660**	**0.0180**	**1.1572**	**1.7499**	**4.7752**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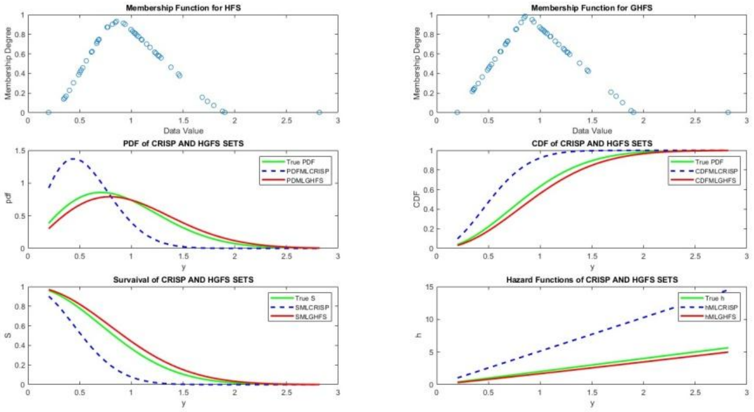
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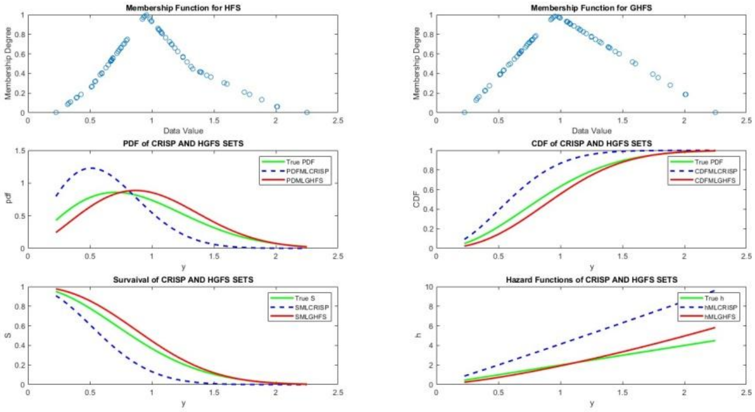
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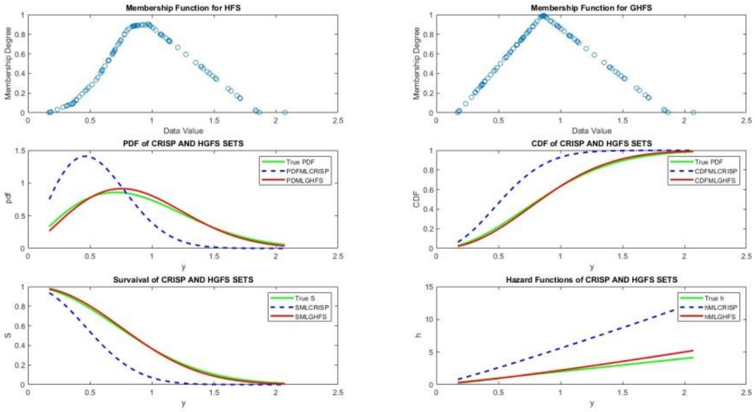
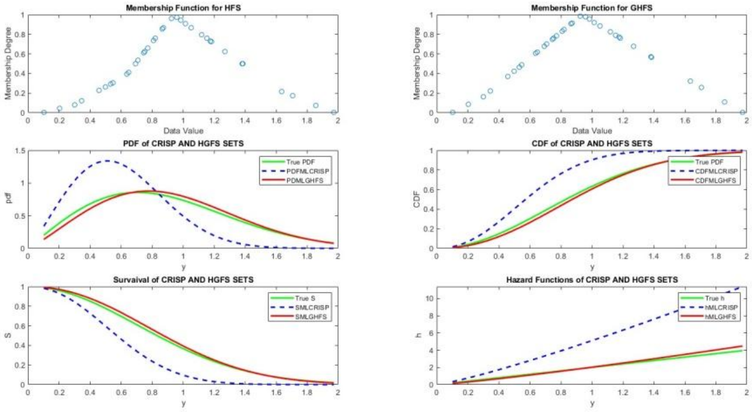


(d)

Figure 1: Membership function curve for crisp and HFGS , PDF, CDF, S and h of curves of Weibull distribution under FSGH set and Crisp set for $\alpha = 1$ and $\lambda = 2$ for all sample sizes and j=3, i=5 under sample sizes a =35, b =50, c =75, d =100

Table 3: Estimated parameters, MSE, MAPE for PDF, CDF, S and h of Weibull distribution under FSGH set and Crisp set for $\alpha = 1$ and $\lambda = 2$ for all sample sizes and j=5, i=3

j	i	n	Criteria Data set	Est. Para.		MSE (PDF)	MAPE (CDF)	MSE (CDF)	MAPE (CDF)	MSE (S)	MAPE (S)	MSE (h)	MAPE (h)
				$\widehat{\alpha}$	$\widehat{\lambda}$								
5	3	35	FSGH	**1.23**	**2.27**	**0.0025**	**8.7449**	**0.0005**	**8.4700**	**0.0005**	**5.3496**	**0.0506**	**9.5991**
			Crisp	**0.68**	**2.28**	**0.1984**	**9.3281**	**0.0307**	**9.3128**	**0.0307**	**6.1775**	**3.6145**	**2.9819**
		50	FSGH	**0.98**	**2.16**	**0.0014**	**3.6788**	**0.0004**	**2.7090**	**0.0010**	**1.9511**	**0.6737**	**1.0860**
			Crisp	**0.59**	**2.38**	**0.1598**	**8.9431**	**0.0200**	**3.8765**	**0.0600**	**2.5006**	**1.6444**	**3.3950**
		75	FSGH	**0.99**	**1.97**	**0.0002**	**2.7303**	**0.0003**	**1.5528**	**0.0006**	**1.6013**	**0.0295**	**1.0418**
			Crisp	**0.47**	**1.77**	**0.1438**	**7.6132**	**0.0190**	**2.6064**	**0.0490**	**2.4697**	**1.4497**	**2.0714**
		100	FSGH	**1.01**	**2.98**	**0.0001**	**1.2860**	**0.0002**	**1.0960**	**0.0001**	**1.3142**	**0.0169**	**1.0182**
			Crisp	**0.30**	**2.56**	**0.1358**	**5.6713**	**0.0111**	**1.2889**	**0.0481**	**2.1459**	**1.1322**	**1.2524**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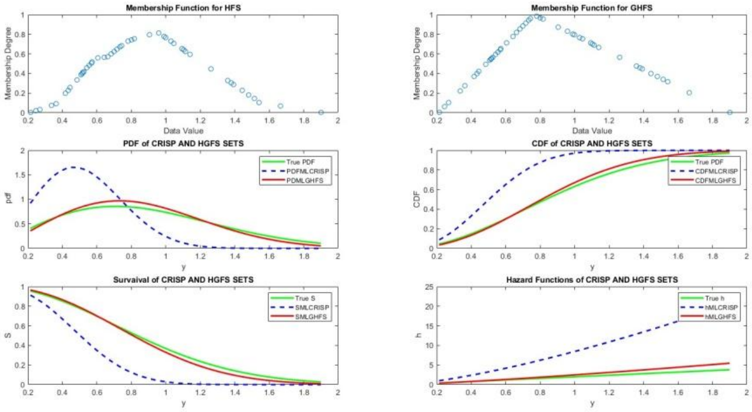
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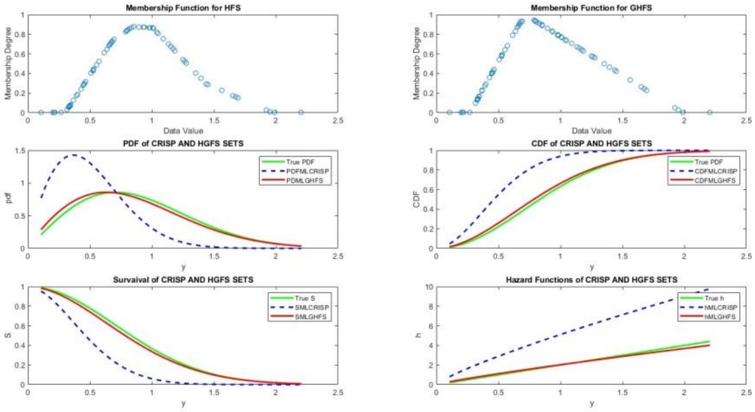
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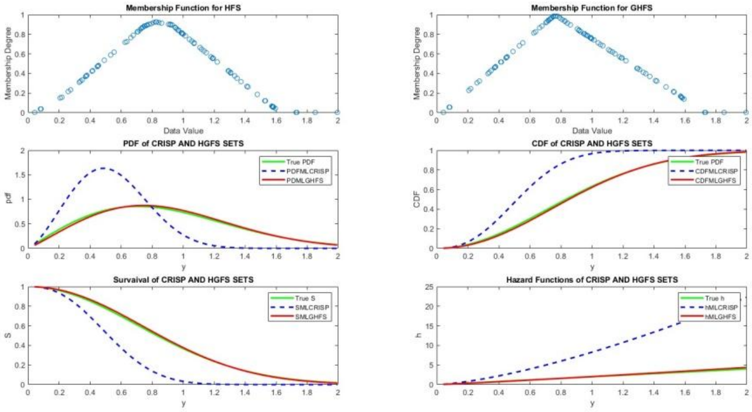

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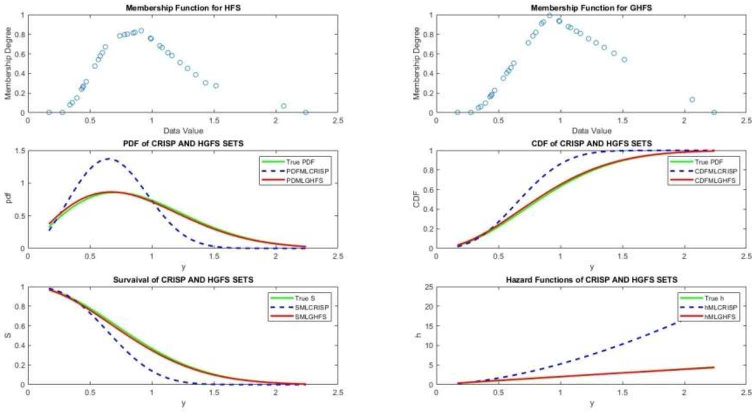
Figure 2: Membership function curve for crisp and HFGS , PDF, CDF, S and h of curves of Weibull distribution under FSGH set and Crisp set for $\alpha = 1$ and $\lambda = 2$ for all sample sizes and j=5, i=3 under sample sizes a=35, b=50, c=75, d=100.

Table 4: Estimated parameters, MSE, MAPE for PDF, CDF, S and h of Weibull distribution under FSGH set and Crisp set for $\alpha = 1$ and $\lambda = 2$ for all sample sizes and j=3, i=10

j	i	n	Criteria Data set	Est. Para.		MSE (PDF)	MAPE (CDF)	MSE (CDF)	MAPE (CDF)	MSE (S)	MAPE (S)	MSE (h)	MAPE (h)
				$\hat{\alpha}$	$\hat{\lambda}$								
3	10	35	FSGH	**0.97**	**1.94**	**0.0009**	**1.3142**	**0.0008**	**1.9235**	**0.0005**	**3.2677**	**0.0034**	**1.4606**
			Crisp	**0.77**	**2.66**	**0.0710**	**7.4194**	**0.0153**	**2.3004**	**0.0153**	**6.5102**	**0.3298**	**4.8603**
		50	FSGH	**1.04**	**2.10**	**0.0006**	**1.1763**	**0.0007**	**0.6765**	**0.0003**	**1.4271**	**0.0026**	**0.8501**
			Crisp	**0.72**	**3.32**	**0.0063**	**2.9296**	**0.0115**	**3.7937**	**0.0315**	**4.9791**	**0.1944**	**1.8915**
		75	FSGH	**0.98**	**1.94**	**0.0005**	**1.0205**	**0.0003**	**0.4297**	**0.0002**	**0.1121**	**0.0018**	**0.3415**
			Crisp	**0.59**	**2.70**	**0.0561**	**1.3542**	**0.0067**	**3.6988**	**0.0227**	**3.9547**	**0.1742**	**1.0828**
		100	FSGH	**1.03**	**1.99**	**0.0003**	**1.0087**	**0.0002**	**0.3099**	**0.0001**	**0.1032**	**0.0010**	**0.0134**
			Crisp	**0.77**	**2.24**	**0.0411**	**0.2563**	**0.0039**	**3.2382**	**0.0239**	**2.6206**	**0.1576**	**1.0526**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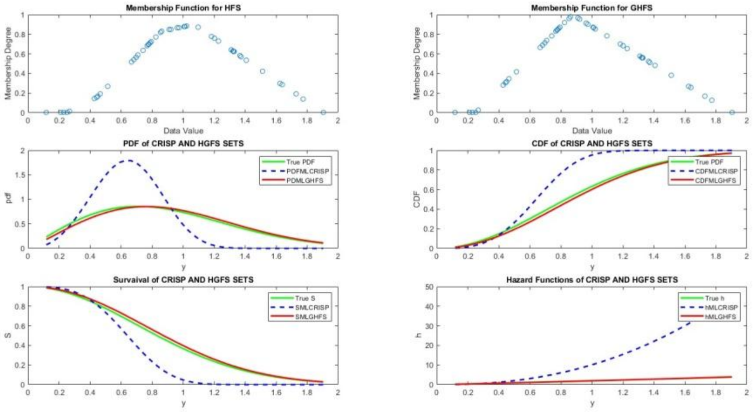
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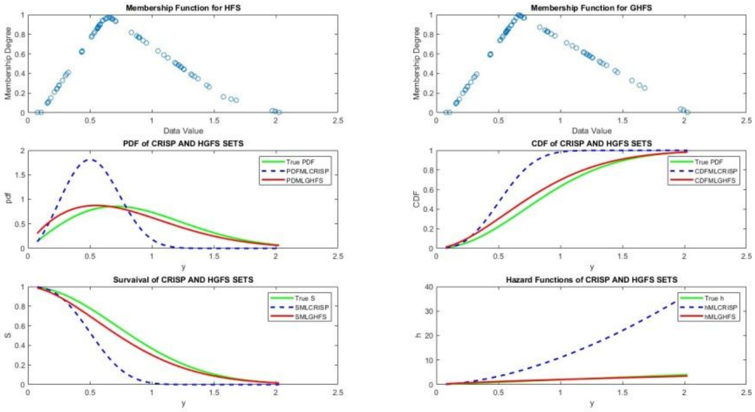
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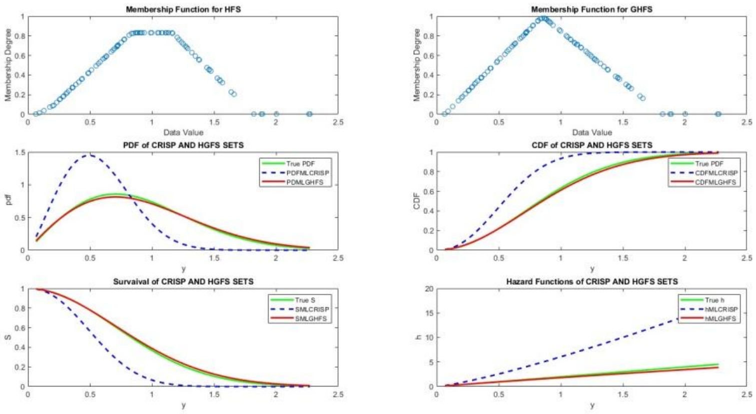

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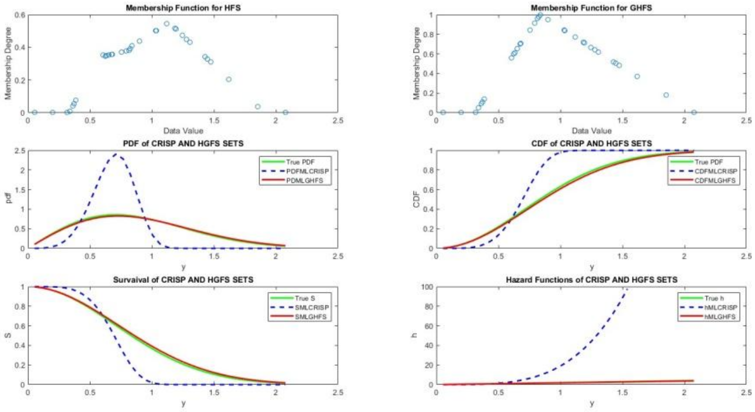
Figure 3: Membership function curve for crisp and HFGS , PDF, CDF, S and h of curves of Weibull distribution under FSGH set and Crisp set for $\alpha = 1$ and $\lambda = 2$ for all sample sizes and j=3, i=10 under sample sizes a=35, b=50, c=75, d=100.

Table 5: Estimated parameters, MSE, MAPE for PDF, CDF, S and h of Weibull distribution under FSGH set and Crisp set for $\alpha = 1$ and $\lambda = 2$ for all sample sizes and j=5, i=5

j	i	n	Criteria Data set	Est. Para.		MSE (PDF)	MAPE (CDF)	MSE (CDF)	MAPE (CDF)	MSE (S)	MAPE (S)	MSE (h)	MAPE (h)
				$\hat{\alpha}$	$\hat{\lambda}$								
5	5	35	FSGH	**1.03**	**1.99**	**0.0005**	**7.2244**	**0.0009**	**2.9380**	**0.0006**	**11.0064**	**0.0322**	**6.5424**
			Crisp	**0.75**	**4.79**	**0.3451**	**15.1115**	**0.0279**	**4.3801**	**9.0279**	**15.3601**	**4.0322**	**5.8429**
		50	FSGH	**0.97**	**1.99**	**0.0004**	**7.0239**	**0.0005**	**0.1334**	**0.0004**	**6.2022**	**0.0037**	**3.3559**
			Crisp	**0.62**	**1.91**	**0.1305**	**14.4049**	**0.0181**	**2.6899**	**4.0381**	**12.7451**	**4.0008**	**3.9138**
		75	FSGH	**1.01**	**2.20**	**0.0003**	**6.1747**	**0.0004**	**0.1049**	**0.0003**	**4.9931**	**0.0028**	**1.3897**
			Crisp	**0.60**	**2.51**	**0.1157**	**12.3169**	**0.0154**	**1.3461**	**4.0554**	**9.8837**	**3.9736**	**3.6747**
		100	FSGH	**1.02**	**2.12**	**0.0001**	**5.7558**	**0.0003**	**0.0547**	**0.0002**	**4.9134**	**0.0002**	**0.6162**

			Crisp	**0.72**	**2.64**	**0.1015**	**10.8246**	**0.0126**	**0.5661**	**3.0226**	**5.9881**	**3.7169**	**2.6675**



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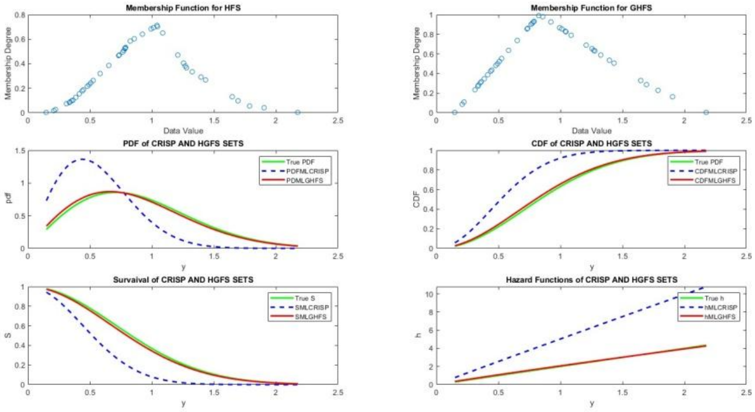

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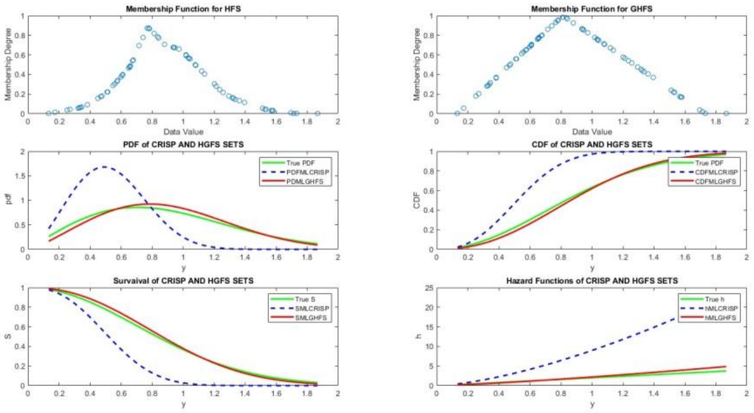

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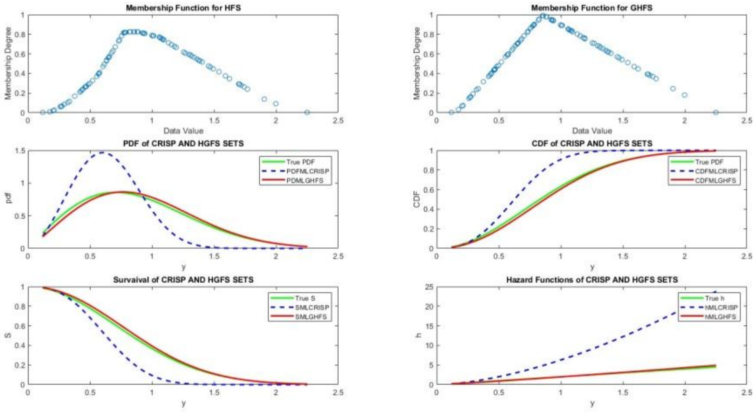

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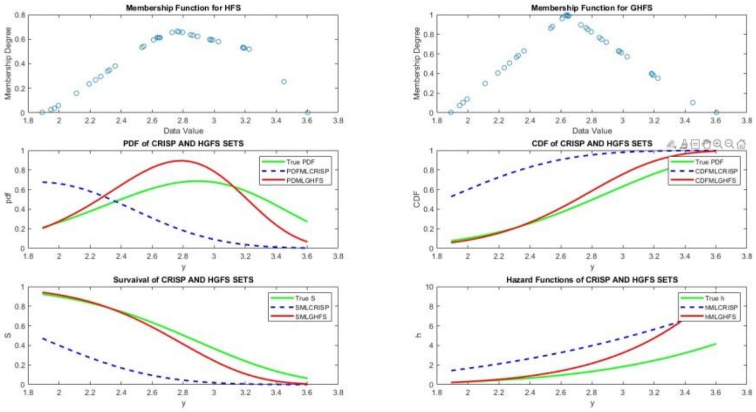
Figure 4: Membership function curve for crisp and HFGS , PDF, CDF, S and h of curves of Weibull distribution under FSGH set and Crisp set for $\alpha = 1$ and $\lambda = 2$ for all sample sizes and j=5, i=5 under sample sizes a=35, b=50, c=75, d=100.

Table 6: Estimated parameters, MSE, MAPE for PDF, CDF, S and h of Weibull distribution under FSGH set and Crisp set for $\alpha = 3$ and $\lambda = 5.5$ for all sample sizes and j=3, i=2

j	i	n	Criteria Data set	Est. Para.		MSE (PDF)	MAPE (CDF)	MSE (CDF)	MAPE (CDF)	MSE (S)	MAPE (S)	MSE (h)	MAPE (h)
				$\hat{\alpha}$	$\hat{\lambda}$								
3	2	35	FSGH	**2.85**	**6.86**	**0.0231**	**26.7806**	**0.0064**	**13.1359**	**0.0051**	**3.2085**	**1.1372**	**4.4913**
			Crisp	**2.05**	**3.61**	**0.1653**	**81.5020**	**0.1842**	**171.0386**	**0.1907**	**9.7125**	**4.6679**	**8.1989**
		50	FSGH	**3.15**	**5.63**	**0.0057**	**22.7648**	**0.0036**	**5.1750**	**0.0043**	**4.3479**	**0.3258**	**4.0258**
			Crisp	**2.07**	**3.19**	**0.1274**	**78.1390**	**0.1310**	**156.1265**	**0.1378**	**8.3618**	**1.5211**	**7.5302**
		75	FSGH	**3.03**	**5.62**	**0.0003**	**6.6103**	**0.0002**	**2.6760**	**0.0007**	**3.0451**	**0.0021**	**3.1048**
			Crisp	**2.10**	**3.91**	**0.1194**	**65.7689**	**0.1241**	**123.6163**	**0.1279**	**7.6466**	**0.3703**	**6.5802**
		100	FSGH	**3.05**	**5.48**	**0.0002**	**2.9009**	**0.0001**	**2.3867**	**0.0005**	**2.5572**	**0.0010**	**3.6271**

			Crisp	**2.18**	**3.76**	**0.1104**	**43.3950**	**0.1032**	**98.5856**	**0.1027**	**6.3254**	**0.2573**	**5.2434**



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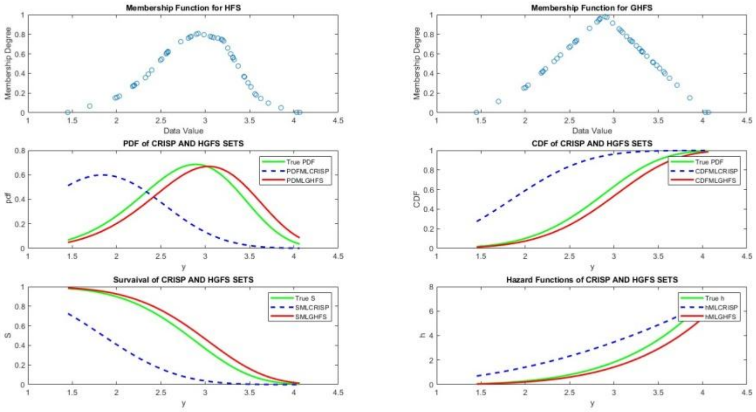

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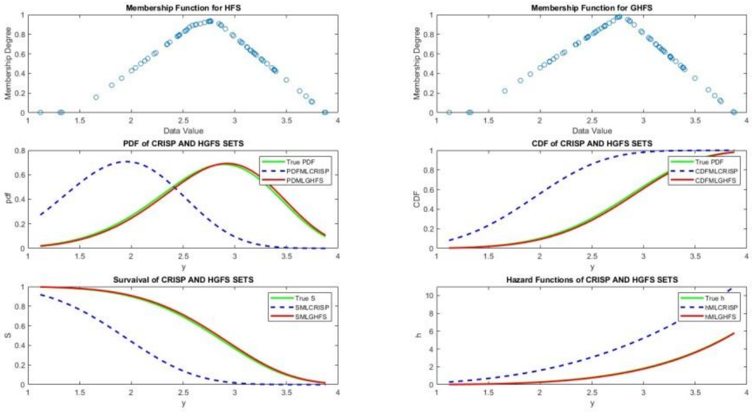

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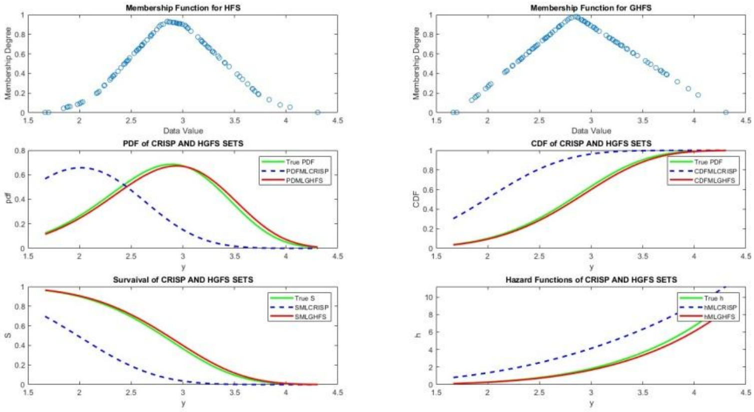

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Figure 5: Membership function curve for crisp and HFGS , PDF, CDF, S and h of curves of Weibull distribution under FSGH set and Crisp set for α = 3 and λ = 5.5 for all sample sizes and j=3, i=2 under sample sizes a=35, b=50, c=75, d=100.

6. Results Discussion

In general, we find that FSGH tends to give estimates closer to the true values of the Weibull distribution parameters. At sample size (35), the parameter estimates were almost far from the theoretical parameter values, since the maximum likelihood method gives inaccurate estimates when the sample size is small, which is consistent with statistical theory. At sample sizes (n=50, 75, 100), we find that the estimates of the distribution parameters tend to closely approach the theoretical parameters in the proposed granular fuzzy hesitating sample, while the parameter estimates under the crisp sample vector tend to move away from the theoretical values. When the number of granules increases, the accuracy of the maximum likelihood estimates increases. In general, when the number of granules and the number of iterations in the GHF sample increases, the accuracy of the estimates increases, and thus the accuracy of the proposed method increases. As the sample size increases, the error measures (MSE, MAPE) of both FSGH and Crisp decrease, but FSGH continues to outperform crisp, indicating that it is more stable and accurate when dealing with large amounts of data.

7. Conclusions

Through the results that have been reached, we can see that the proposed FSGH is better than the crisp data set for the Weibull distribution, and that the method achieves the lowest error criteria, especially in large data sets. And that with increasing hesitation and granularity, the accuracy of the method increases, and therefore we recommend using it in the case of problems that require complex and ambiguous decisions.

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