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Article

Comparative Study to Find the Square Roots of Complex Numbers

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Abstract: Complex numbers have been a cornerstone in mathematics since their inception in the 16th century. Despite their wide-ranging applications in physics, engineering, and other sciences, solving quadratic equations and finding the square roots of complex numbers remain challenging. This study introduces a novel method for finding square roots of complex numbers and compares its effectiveness with De Moivre's theorem. The research applies De Moivre's theorem to solve complex equations and derives a new analytical method for simplifying these solutions. The study evaluates both methods by solving a series of quadratic equations and compares the results in terms of accuracy, efficiency, and computational simplicity. The new method provided solutions identical to those derived using De Moivre's theorem but with a simplified computational process. It was observed that the proposed method reduces complexity, minimizes computational steps, and achieves faster results without sacrificing accuracy. The novel analytical method is a robust alternative to De Moivre's theorem for solving complex quadratic equations and finding square roots. It demonstrates significant advantages in terms of simplicity and speed, making it a valuable tool for mathematical and practical applications. This approach offers a streamlined process for addressing complex problems, bridging theoretical and practical insights in mathematical problem-solving.

Keywords: As a Result of De Moivre's Theorem , Num, erical Questions , a New Method for Finding Square Roots in a Simplified Manner

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1. Introduction

In 1545 complex numbers appeared primitively by the Italian scientist Giro Lamo Cardano when he used the solution of the equation of the third degree as the scientist Raphael Yumbelli worked in this field and developed by the scientist Leonhart Euler and others in 1748 [2] . A complex number is a number that can be written in the form ($a + b\,i$) (a) is the real number and (b) is the imaginary number as it is called in the regular form and (a , b) in the Cartesian form [14].

Complex numbers have a great place in mathematics, and they also play an important role in various scientific applications [3]. Complex numbers are used in the fields of electricity, dynamics, the theory of relativity , and almost all areas of physics [1]., but the set of complex numbers is the most difficult to understand because it includes the imaginary numbers, Complex numbers do not exist in nature like negative numbers, as there is a difference between the sciences that depend on reality, which are the human and natural sciences, and between the sciences of mathematics that are related to the mind and its vast imaginative capabilities , where the mind can link those fantasies in a logical and sound connection that does not contradict it , so the numbers Vehicle and most mathematics belong to the area of mental imagination. Properties of a complex number include addition, subtraction, and division , $\overline{C}$ is also called the conjugate of a complex number, which in turn changes the sign of an imaginary number , such as $\overline{(a+bi)} = (a-bi)$ It is also distributed among the four processes [5 , 6 , 7 , 9 , 10].

2. Materials and Methods

Apply De Moivre's theorem to find the square roots of complex numbers [5 , 12]

So be it $M_1 = cos\theta + isin\theta$ (1) and $M_2 = cos\emptyset + isin\emptyset$ (2) using M_1 *and* M_2 the polar formula :

$$M_1 \times M_2 = (\cos(\theta) + isin(\theta))(\cos(\emptyset) + isin(\emptyset))$$
$$= \cos(\theta)\cos(\emptyset) + \cos(\theta)\, isin(\emptyset) + isin(\theta)\cos(\emptyset) + i^2\sin(\theta)\sin(\emptyset)$$
$$= (\cos(\theta)\cos(\emptyset) - \sin(\theta)\sin(\emptyset)) + i(\cos(\theta)\sin(\emptyset) + \sin(\theta)\cos(\emptyset))$$
$$= \cos((\theta) + (\emptyset)) + isin((\theta) + (\emptyset)) \quad (3)$$

From equation (3) if we assume that ($\emptyset = \theta$) we deduce the following relation

$$(\cos(\theta) + i\sin(\theta))^2 = \cos(2\theta) + i\sin(2\theta) \quad (4)$$

It can be shown that the first side of equation (4) is equal to the second side as follows:

$$(\cos(\theta) + i\sin(\theta))^2 = cos^2(\theta) + 2\, i\cos(\theta)\sin(\theta) - sin^2(\theta)$$
$$= (cos^2(\theta) - sin^2(\theta)) + i\, 2(\cos(\theta)\sin(\theta))$$
$$= \cos(2\theta) + i\sin(2\theta)$$

(1664-1754) This relationship was popularized as De Moivre's theorem and can be proven by the method of mathematical stability [11] .

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta) \quad , \theta \in R \quad , n \in N \quad (5)$$

We consider = 1, the relation becomes

$(\cos(\theta) + i\sin(\theta))^1 = \cos(1\theta) + i\sin(1\theta)$ This is correct

Let's take k ≥ 1 and assume that the relation is true for each n= k ie, n:

$(\cos(\theta) + i\sin(\theta))^k = \cos(k\theta) + i\sin(k\theta)$, correct .

Here the relationship is valid n=k+1:

$$(\cos(\theta) + i\sin(\theta))^{k+1} = (\cos(\theta) + isin(\theta))^k(\cos(\theta) + i\sin(\theta))^1$$
$$= (\cos(k\theta) + i\sin(k\theta))(\cos(\theta) + i\sin(\theta))$$
$$= cos((k) + (K\theta)) + i\sin((k) + (k\theta))$$
$$= cos(k+1)(\theta) + i\sin(k+1)(\theta)$$

In fact, the relationship is correct $n = k$, $k \geq 1$ They are correct correctly $n = k + 1$

By mathematical stability, the theorem is valid for all values n .

Apply the theorem to dictate the quadratic equation [3 , 8 , 13]

Consider that an ordinary quadratic equation has the form :

$$Q^2 = \mathcal{M} \pm \aleph\, i \quad (6)$$

When $Q = s + di$ and $\mathcal{M}$ The real part and $\aleph$ the imaginary part

When we root both sides of equation (6), we get: $Q = \sqrt{\mathcal{M} \pm \aleph\, i}$ (7)

From equation (a) we find (Modulus of Complex Number) ,, (mod Q) or $\|Q\|$

$$mod(Q) = \sqrt{\mathcal{M}^2 + \aleph^2} = \ell \quad (8)$$

Then we find (Argument of Complex Number) , $\theta = \arg(Q)$ through:

$$sin\theta = \frac{\aleph}{modQ} \quad , \quad cos\theta = \frac{\mathcal{M}}{modQ} \quad (9)$$

With equations (5 , 8 ,9) we derive the relation:

$$Q = \ell\,(cos\theta + i\, sin\theta) \quad (10)$$

By adding a root to both sides eq (10) , we conclude

$$\sqrt[n]{Q} = \sqrt[n]{\ell}\left[cos\frac{\theta+2\pi\rho}{n} + isin\frac{\theta+2\pi\rho}{n}\right] , \theta \in R , n \in Z^+ , \rho = 0,1, \dots\dots, n-1 \quad (11)$$

(11) This relationship is defined as a consequence of De Moivre's theorem

Analysis of the new complex method [8]

Using equations (6 and 8) we can deduce the relationship. To find the square roots of equation (8) directly and easily using the formula:

$$\sqrt{\mathcal{M} \pm \aleph\, i} = \pm \left[\sqrt{\frac{1}{2}\left(\sqrt{\mathcal{M}^2 + \aleph^2} + \mathcal{M}\right)} \;\pm\; i\,\sqrt{\frac{1}{2}\left(\sqrt{\mathcal{M}^2 + \aleph^2} - \mathcal{M}\right)} \right] \quad (12)$$

Substituting equation (8) into equation (6), we get:

$$\sqrt{\mathcal{M} \pm \aleph\, i} = \pm \left[\sqrt{\frac{1}{2}\left(\ell + \mathcal{M}\right)} \;\pm\; i\,\sqrt{\frac{1}{2}\left(\ell - \mathcal{M}\right)} \right] \quad (13)$$

Thus, equation (6) is a solution to equation (13) in the set of complex numbers .

3. Results and Discussion

Practical applications of the new complex method analysis and comparison of the solution with the result of De Moivre's theorem

Solve the equation in complex numbers

$$\varkappa = \sqrt{1 + \sqrt{3}\, i} \quad (14)$$

Using De Moivre and using equations (8,9,10), we obtain

$$mod(\varkappa) = 2 \quad (15)$$

We also find the capacitance of the compound through:

$$c\,os\theta = \frac{1}{2} \quad and \quad sin\theta = \frac{\sqrt{3}}{2} \quad (16)$$

We deduce the angle in the first quadrant, namely $\arg(\varkappa) = \frac{\pi}{3}$ (17)

And using De Moivre's formula

$$\mho = mod(\varkappa)[cosarg(\varkappa) + isin\,arg(\varkappa)] \quad (18)$$

$$\varkappa = 2\left[cos\frac{\pi}{3} + isin\frac{\pi}{3}\right] \quad (19)$$

From equation (11) we find

$$\sqrt{\varkappa} = \sqrt{2}\left[cos\frac{\frac{\pi}{3}+2\pi\rho}{2} + isin\frac{\frac{\pi}{3}+2\pi\rho}{2}\right], \rho = 0,1 \quad (20)$$

When ρ = 0 We get

$$\varkappa_1 = \sqrt{2}\left[os\frac{\pi}{6} + isin\frac{\pi}{6}\right]$$
$$= \sqrt{2}\left[\frac{\sqrt{3}}{2} + \frac{1}{2}\,i\right]$$
$$= \frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i \quad (21)$$

When ρ = 1 We get:

$$\varkappa_2 = \sqrt{2}\left[os\frac{7\pi}{6} + isin\frac{7\pi}{6}\right]$$
$$= \sqrt{2}\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}\,i\right]$$
$$= -\frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\,i \quad (22)$$

From equations (21, 22), the solution set of equation (14) is represented using De Moivre's result theorem (generalization).

We will discuss the solution of equation (14) by (new complex method analysis) and compare the results with De Moivre's result theorem:

Using equation (12) we get :

$$\sqrt{\varkappa} = \pm \left[\sqrt{\frac{1}{2}\left(\sqrt{(1)^2 + \left(\sqrt{3}\right)^2} + 1\right)} + i\,\sqrt{\frac{1}{2}\left(\sqrt{(1)^2 + \left(\sqrt{3}\right)^2} - 1\right)} \right]$$

Using equation (13), we get:

$$\varkappa_t = \pm \left[\sqrt{\frac{1}{2}\left(\sqrt{4} + 1\right)} + i\,\sqrt{\frac{1}{2}\left(\sqrt{4} - 1\right)} \right] \text{ When t=1 ,2}$$

$$= \pm \left[\sqrt{\frac{1}{2}(2+1)} + i\sqrt{\frac{1}{2}(2-1)} \right]$$

$$= \pm \left[\sqrt{\frac{3}{2}} + i\sqrt{\frac{1}{2}} \right]$$

$$= \pm \left[\frac{\sqrt{3}}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \right]$$

When $t = 1$ We get:

$$\varkappa_1 = \frac{\sqrt{3}}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \quad (22)$$

When $t = 2$ We get:

$$\varkappa_2 = -\frac{\sqrt{3}}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \quad (23)$$

It was observed that equations (22 and 23) as a solution to equation (14) according to the (**Analysis of the new complex method**) method are identical to the solution to equations (21, 22) according to the De Moover method ..

Find the square roots of the complex equation using De Moivre's theorem and compare the results with the new complex method analysis

$$e^2 = -1 \quad (24)$$

Equation (24) can be solved using De Moivre's theorem and using the formulas (7 , 8 ,9 , 10) we get the relationship:

$$e^2 = (cos\pi + isin\pi) \quad (25)$$

Through the square root of both sides of equation (25) We get :

$$e = \sqrt{(cos\pi + isin\pi)} \quad (26)$$

$$e = (cos\pi + isin\pi)^{\frac{1}{2}}$$

We will go through using equation (11) which we get :

$$e = \left[cos\frac{\pi+2\pi\rho}{2} + isin\frac{\pi+2\pi\rho}{2}\right] \quad , \rho = 0,1 \quad (27\ )$$

: We get $\rho = 0$ When

$$e_1 = cos\frac{\pi}{2} + isin\frac{\pi}{2} = 0 + i \quad (28)$$

: We get $\rho = 1$ When

$$e_2 = cos\frac{3\pi}{2} + isin\frac{3\pi}{2} = 0 - i \quad (29)$$

The equations (8 , 29) are a solution to the complex equation (24) using the generalization theorem [20].

Now (24) using the **(new complex analysis method**) and compare the solutions with the generalization theorem.

Using equation (8, 12, 13) we get

$$e_w = \pm \left[\sqrt{\frac{1}{2}\left(\sqrt{(0)^2 + (1)^2} + (-1)\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{(0)^2 + (1)^2} - (-1)\right)} \right]$$

$w = 1,2$: When

$$e_w = \pm [\ 0 + i\ ]$$

We get : $w = 1$ When

$$e_1 = 0 + i \quad (30)$$

We get : $w = 2$ When

$$e_2 = 0 - i \quad (31)$$

Equations (30 and 31) represent the roots of equation (24) by (the new complex analysis method) and it has been observed that the solutions are identical to the generalization theory, but in a simplified manner and faster steps [19]

Comparing the answers of the quadratic equation between the generalization theory and (the new complex analysis method)

$$O^2 - \left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = 0 \qquad (32)$$

The complex equation (32) will be solved using the result of De Moivre's (generalizing) theorem by equation (11), and we will get :

$$O^2 = \left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) \qquad (33)$$

Entering the square root of both sides of the equation (33), we get the relation

$$O = \left[cos\frac{\frac{\pi}{3}+2\pi\rho}{2} + isin\frac{\frac{\pi}{3}+2\pi\rho}{2}\right] \quad , \ \theta \in R \,, n \in Z^+ \ , \ \rho = 0,1 \qquad (34)$$

When we substitute $\rho = 0,1$ in (34) we get the following roots

We get $\rho = 0$ When

$$O_1 = \left[cos\frac{\pi}{6} + isin\frac{\pi}{6}\right]$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\,i \qquad (35)$$

When $\rho = 1$ We get:

$$O_2 = \left[cos\frac{7\pi}{6} + isin\frac{7\pi}{6}\right]$$

$$= -\frac{\sqrt{3}}{2} - \frac{1}{2}\,i \qquad (36)$$

We notice that equations (35 and 36) represent a radical solution to equation (33). Equation (33) will be solved using (new complex analysis method)

Comparison of results with theory (generalization)

We use relation (12 ,13) to find solutions to equation (32

$$O = \pm\left[\sqrt{\frac{1}{2}\left(\sqrt{\left(cos\frac{\pi}{3}\right)^2 + \left(sin\frac{\pi}{3}\right)^2} + cos\frac{\pi}{3}\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{\left(cos\frac{\pi}{3}\right)^2 + \left(sin\frac{\pi}{3}\right)^2} - cos\frac{\pi}{3}\right)}\right]$$

$$O = \pm\left[\sqrt{\frac{1}{2}\left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2}\right)}\right]$$

$$O = \pm\left[\sqrt{\frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{3}{4}} + \frac{1}{2}\right)} + i\sqrt{\frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{3}{4}} - \frac{1}{2}\right)}\right]$$

$$O = \pm\left[\sqrt{\frac{3}{4}} + i\sqrt{\frac{1}{4}}\right] \qquad (37)$$

From equation (37) we get the following roots:

$$O_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}\,i \qquad (38)$$

$$O_2 = -\frac{\sqrt{3}}{2} - \frac{1}{2}\,i \qquad (39)$$

The equations (38 , 39) are considered a solution to equation (35,36) using the (new complex analysis method) . It is observed that the solution matches the equations (33) using the generalization theorem [18].

4. Conclusion

In this paper , a new method for solving complex quadratic equations defined by **(Analysis of the new complex method**) will be defined and its results will be compared with the result of De Moivre's theory (generalization) [15]. It has also been noted that (**Analysis of the new complex method**) is an effective method for finding square roots and for solving complex quadratic equations. Where it is considered easy to apply and steps , and it has strong results, and it gives results very quickly [16]. It is almost more accurate than the result of De Moivre's theorem in finding the square roots of a complex number. Also, (**Analysis of the new complex method**) is able to transform the most difficult problems into a simple, easy and understandable solution that is free from difficulties and impurities [17].

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