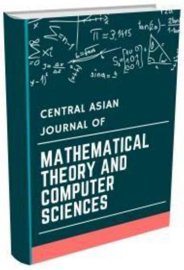

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Article

Study of the Mixture Distribution of the Lomax-Weibull Equation and Its Statistical Properties

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Abstract: In order to overcome the drawbacks of traditional probability distributions for simulating real-world data, this study introduces a novel mixed distribution model called the Lomax-Weibull distribution (LWD). Although several mixed distributions have been studied in previous studies, a thorough examination of LWD's statistical characteristics is still missing. By analyzing its probability density, survival, and hazard functions, this research closes that gap. Model parameters were computed using the maximum likelihood estimation (MLE) approach. The findings demonstrate that LWD offers a more adaptable tool for simulating intricate data patterns, which may find use in disciplines like reliability engineering and actuarial science. These discoveries enhance statistical models for phenomena that occur in the real world.

Keywords: Lomax distribution, Weibull distribution, Moment, MLE, Order statistic, mixture of distributions

1. Introduction

In some cases, classical probability distributions do not fit the phenomenon under study and to improve the fit of the data to the distribution, new families of distributions, namely mixed distributions, have been created. Therefore, mixed distributions will be obtained by adding one or more parameters to the distribution to make these distributions more flexible for practical application to real data.

Newly created families of continuous distributions have drawn the attention of various statisticians in recent years, encouraging them to create novel models. By adding one or more extra shape parameter(s) to the baseline distribution, Among the families that were created are**:** the Weibull-Exponential [1], and Weibull-Linduly distribution [2], Edwin Cordeiro, Ortega, Popove, and others [3] examined and analyzed the Gamma-Lomax distribution and its characteristics. And [4] suggested an exponentiated beta with five parameters. Lomax arrangement. In their study, [5] employed a three-parameter Power Lomax distribution to predict bladder cancer patients' remission durations. [6] applied the Gompertz–Lomax distribution with increasing, decreasing, and constant failure rate to data on the strengths of 1.5 cm glass fibers. Regarding the 5-Parameter Lomax Distribution: Characteristics and Uses [4]. An investigation into the use of the Halflogistic Lomax distribution with patient data related to bladder cancer was conducted by [7], a novel applica-

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tion of the T-X method to the Maxwell and Lomex distributions is suggested and examined. Moments and other structural and statistical features of the suggested distribution are found [8].

The transmuted Exponential Lomax distribution [9], propose a model for probability distribution Power Function-Truncated Burr III [8], a brand-new distribution combination known as Power Function-Truncated Burr III (MPF-TBIII) [10], a heavy-tailed alternative to exponential, Weibull, gamma distributions, the Lomax distribution was invented by [11] and described by [12]. It has gained popularity in the literature on distribution theory and has a wide range of applications in actuarial science, income and wealth inequality, engineering, medical and biological sciences, engineering, city size, and reliability modeling. originates as a limiting distribution of residual lifetimes at large age in the lifetime context (see Balkema and deHann, 1974), [13], According to Chahkandi (2009), the Lomax model is a member of the family of models with decreasing failure rates. See [14] for additional details on the Pareto family and Lomax distribution. The exponentiated Lomax and the Lomax model are among the family of decreasing failure rate generalizations of the Lomax distribution that have been explored [15].

Other significant research on the Lomax distribution and its variations was conducted by [16], who examined four different approaches for calculating the Lomax distribution's parameter, The Weibull distribution is considered one of the most famous families of failure distributions. It was derived by the Swedish scientist (Weibull 1939). He showed some applications of the distribution. The Weibull distribution was used to describe the distribution in failure cases and in describing the failure of some electrical devices, as the probability density function of this distribution can be derived from the concept of the risk rate [17]. This study looked at the distribution characteristics, survival function, hazard function, ordered statistics, and central moments after deriving the Lomax-Weibull distribution mixture. Additionally, the maximum likelihood estimator method were used to estimate the distribution coefficients.

2. Materials and Methods

The Lomax Distribution's probability density function is defined as follows:

$$f_1(x) = \lambda k(1+\lambda x)^{-(k+1)} \quad ; x > 0, \lambda, k > 0 \qquad (1)$$

Such that λ shape parameter and k scale parameter. The Cumulative distribution function defined by:

$$F_1(x) = 1-(1+\lambda x)^{-k} \qquad (2)$$

The Lomax distribution's Reliability function $R_1(x)$ is defined by:

$$R_1(x) = (1+\lambda x)^{-k} \qquad (3)$$

and the hazard function

$$h_1(x) = \frac{f_1(x)}{R_1(x)} = \frac{\lambda k}{(1+\lambda x)} \qquad (4)$$

The (PDF) of the Weibull Distribution defined by:

$$f_2(x) = \beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta} \quad ; x, \beta, \alpha > 0 \qquad (5)$$

the cumulative distribution function provided by:

$$F_2(x) = (1 - \exp\left(\frac{-x}{\alpha}\right)^{\beta}) \quad (6)$$

where, α shape parameter and β scale parameter,

The Weibull distribution's Reliability function $R_2(x)$ is defined as:

$$R_2(x) = \exp\left(\frac{-x}{\alpha}\right)^{\beta} \quad (7)$$

and the hazard function:

$$h_2(x) = \frac{f_2(x)}{R_2(x)} = \frac{\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}}{\exp\left(\frac{-x}{\alpha}\right)^{\beta}} = \alpha^{-\beta}x^{\beta-1} \quad (8)$$

3. Results and Discussion

The Lomax Weibull Distribution (LWD)

The Lomax-Weibull's probability distribution function, as provided by:

$$f_{LW}(x) = zf_1(x) + (1-z)f_2(x) \quad (9)$$

Where $f_1(x)$is (pdf) of Lomax distribution and $f_2(x)$ is the (pdf) of weibull distribution, and z is the Mixing proportion parameter and let z= a/1+a such that:

$$f_{LW}(x;\psi) = \frac{a}{1+a}\left(\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right) + \frac{1}{1+a}\left(\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}\right); x,\psi > 0 \quad (10)$$

When $\psi = \lambda, k, \alpha, \beta, a$, and we notice that both conditions are met.

$$f_{LW}(x;\psi) > 0 \quad , \quad \int_0^{\infty} f_{LW}(x;\psi)\,dx = 1$$

$$\int_0^{\infty} f_{LW}(x;\psi)\,dx = \int_0^{\infty} \frac{a}{1+a}\left(\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right) + \frac{1}{1+a}\left(\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}\right)dx$$

$$let\ v_1 = \int_0^{\infty} \frac{a}{1+a}\left(\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right)dx = \frac{a}{1+a}$$

$$let\ v_2 = \int_0^{\infty} \frac{1}{1+a}\left(\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}\right)dx = \frac{1}{1+a}$$

$$now\ v_1 + v_2 = \frac{a}{1+a} + \frac{1}{1+a} = 1$$

The cumulative distribution function:

$$F_{LW}(x;\psi) = \int_0^{x} f_{LW}(x;\psi)\,dx$$

$$F_{LW}(x;\psi) = \frac{a}{1+a}\,(1-(1+\lambda x)^{-k}) + \frac{1}{1+a}\left(1-\exp\left(\frac{-x}{\alpha}\right)^{\beta}\right) \quad (11)$$

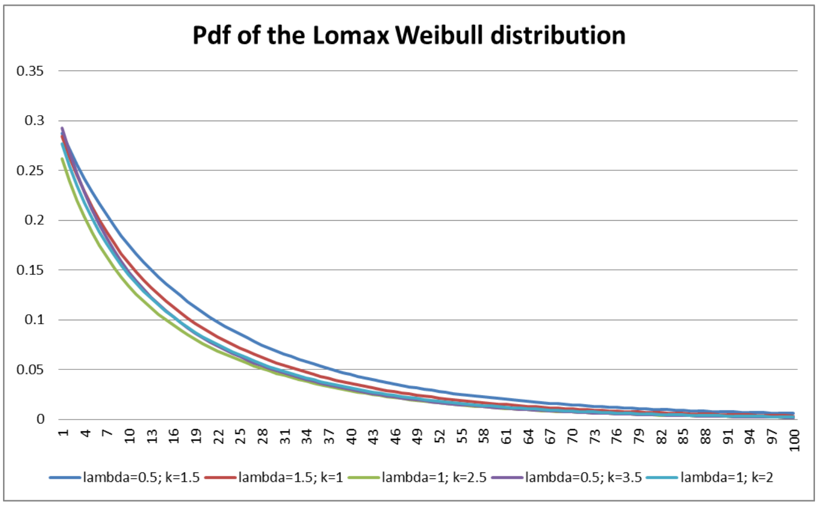

Figure 1. Pdf of the (LWD) when α=2, and β=a=1

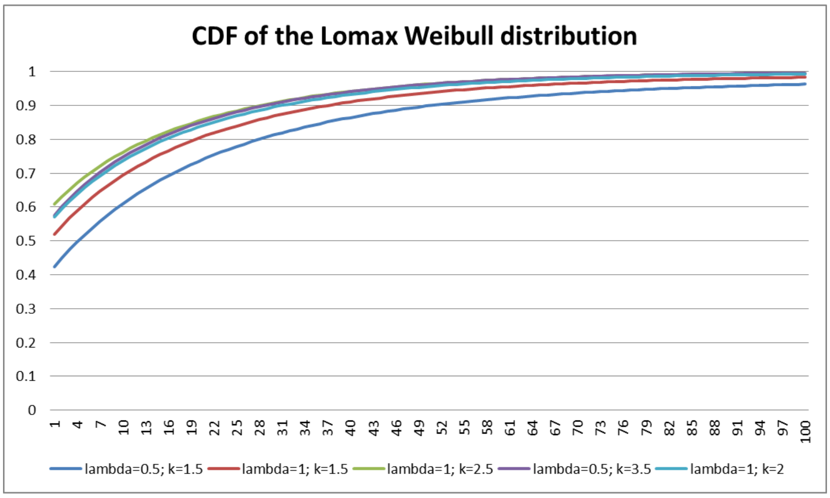

Figure 2. CDF of the (LWD) when α=2, and β=a=1

The Survival function (Reliability) and hazard function:

The survival function (Reliability) of the proposed distribution provided by:

$$S_{LW}(x) = \frac{a(1+\lambda x)^{-k} + \exp\left(\frac{-x}{\alpha}\right)^{\beta}}{1+a} \quad (12)$$

The hazard function is defined as:

$$h_{LW}(x) = \frac{f_{LW}(x)}{S_{LW}(x)} = \frac{a\left(\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right) + \beta \alpha^{-\beta} x^{\beta-1} \exp\left(\frac{-x}{\alpha}\right)^{\beta}}{a(1+\lambda x)^{-k} + \exp\left(\frac{-x}{\alpha}\right)^{\beta}} \quad (13)$$

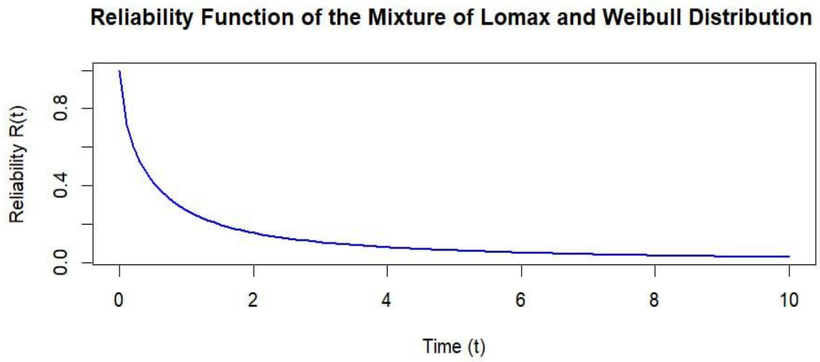

Figure 3. Reliability function of (LW) distribution

r-th moment of mixture distribution

- Non central moments:

$$E(x^r) = \int_0^\infty x^r \left[\frac{a}{1+a}\left(\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right) + \frac{1}{1+a}\left(\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}\right)\right]dx \quad (14)$$

$$Let\ p_1 = \int_0^\infty x^r \left[\frac{a}{1+a}\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right]dx$$

$$suppose\ that\ y_1 = 1+\lambda x\ ,\ x = \frac{y_1-1}{\lambda}\quad ,dx = \frac{dy_1}{\lambda}$$

get to

$$p_1 = \frac{a}{1+a}k\lambda^{-r}\int_0^\infty (y_1-1)^r y_1^{-(k+1)}dy_1 = \frac{a}{1+a}\lambda^{-r}\frac{\Gamma(k-r)\Gamma(r+1)}{\Gamma k} \quad (15)$$

$$Let\ \ p_2 = \frac{1}{1+a}\int_0^\infty x^r[\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}]\,dx$$

$$suppose\ that\ \ y_2 = \left(\frac{x}{\alpha}\right)^{\beta},\ x = \alpha y_2^{\frac{1}{\beta}},\quad \frac{\alpha}{\beta}y_2^{\frac{1}{\beta}-1}$$

get to

$$p_2 = \frac{1}{1+a}\alpha^r\int_0^\infty y_2^{\frac{r}{\beta}}\exp(-y_2)\,dy_2 = \frac{\alpha^r\Gamma(1+\frac{r}{\beta})}{\beta(1+a)} \quad (16)$$

$$now\ \ E(x^r) = p_1 + p_2$$

$$E(x^r) = \frac{a\lambda^{-r}\Gamma(k-r)\Gamma(r+1)}{(1+a)\Gamma k} + \frac{\alpha^r\Gamma(1+\frac{r}{\beta})}{\beta(1+a)} \quad (17)$$

If r=1

$$\mu = E(x) = \frac{a}{\lambda k(1+a)} + \frac{\alpha\Gamma(1+\frac{1}{\beta})}{\beta(1+a)} \quad (18)$$

If r=2

$$E(x^2) = \frac{2a}{\lambda^2 k(k+1)(1+a)} + \frac{\alpha^2\Gamma(1+\frac{2}{\beta})}{\beta(1+a)} \quad (19)$$

Now the variance defined by:

$$v(x) = E(x^2) - \left(E(x)\right)^2 \quad (20)$$

- Central moments:

$$\mu_r = E\left(x - E(x)\right)^r$$

$$\mu_r = E\left(\sum_{i=0}^{r} C_i^r (-1)^i x^{r-i} \mu_1^i\right) \quad (21)$$

Order statistic

Let x_1, x_2, ..., x_n are sample with size n and has Lomax Weibull distribution with probability dinsity function $f_{LW}(x)$ and let $x_{1;n} \leq x_{2;n} \leq \ldots \leq x_{r;n} \ldots \leq x_{n;n}$ are order statistic, The joint probability density function for the ordered statistics is provided by:

$$w_{r,n}(x) = \frac{n!}{(r-1)!(n-r)!}\left(W(x)\right)^{r-1}\left(1 - W(x)\right)^{n-r} w(x)$$

$$w_{r,n}(x) = \frac{n!}{(r-1)!(n-r)!}\left[\frac{a}{1+a}\left(1-(1+\lambda x)^{-k}\right) + \frac{1}{1+a}\left(1 - \exp\left(\frac{-x}{\alpha}\right)^{\beta}\right)\right]^{r-1} * \left(1 - \left(\frac{a}{1+a}\left(1-(1+\lambda x)^{-k}\right) + \frac{1}{1+a}\left(1 - \exp\left(\frac{-x}{\alpha}\right)^{\beta}\right)\right)\right)^{n-r} * \left[\frac{a}{1+a}\left(\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right) + \frac{1}{1+a}\left(\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}\right)\right] \quad (22)$$

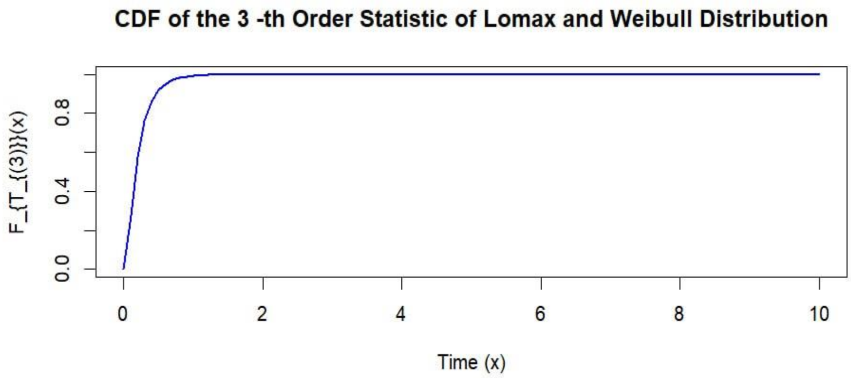

Figure 4. CDF of Order Statistic for the (LWD)

Parameters Estimation of (LW) Distribution

Maximum Likelihood Estimation With unknown parameters α, β, λ, k, and a specified previously, let x_1, x_2, ..., x_n be a sample of size n independently and identically distributed random variables from the (LWD).

The Likelihood function is define by:

$$L\left(f_{LW}(x;\psi)\right) = \prod_{i=1}^{n} f_{LW}(x;\psi) \quad (23)$$

$$Ln\left(Lf_{LW}(x;\psi)\right) = \sum_{i=1}^{n} Ln\left(\frac{a}{1+a}\left(\frac{\lambda k}{(1+\lambda x)^{(k+1)}}\right) + \frac{1}{1+a}\left(\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}\right)\right) \quad (24)$$

$$Ln\left(Lf_{LW}(x;\psi)\right) = \sum_{i=1}^{n} Ln[\frac{a\lambda k(1+\lambda x_i)^{-(k+1)}+\beta\alpha^{-\beta}x^{\beta-1}\exp\left(\frac{-x}{\alpha}\right)^{\beta}}{1+a} \quad (25)$$

The following nonlinear equations result from computing the first-order partial derivatives of (25) with regard to α, β, λ, k, and a and equating to zero:

$$\frac{\partial(Ln(L)f_{LW}(x;\psi))}{\partial(a)} = \sum_{i=1}^{n} \frac{\lambda k(1+\lambda x_i)^{-(k+1)}+\beta\alpha^{-\beta}x_i{}^{\beta-1}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}}{[a\lambda k(1+\lambda x_i)^{-(k+1)}+\beta\alpha^{-\beta}x_i{}^{\beta-1}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}]} = 0 \quad (26)$$

$$\frac{\partial(Ln(L)f_{LW}(x;\psi))}{\partial(\lambda)} = \sum_{i=1}^{n} \frac{ak(1+\lambda x_i)^{k}[x_i\lambda(k+1)+(1+\lambda x_i)]}{[a\lambda k(1+\lambda x_i)^{-(k+1)}+\beta\alpha^{-\beta}x_i{}^{\beta-1}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}]} = 0 \quad (27)$$

$$\frac{\partial(Ln(L)f_{LW}(x;\psi))}{\partial(k)} = \sum_{i=1}^{n} \frac{a\lambda k(1+\lambda x_i)^{(k+1)}[Ln(1+\lambda x_i)+1]}{[a\lambda k(1+\lambda x_i)^{-(k+1)}+\beta\alpha^{-\beta}x_i{}^{\beta-1}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}]} = 0 \quad (28)$$

$$\frac{\partial(Ln(L)f_{LW}(x;\psi))}{\partial(\alpha)} = \sum_{i=1}^{n} \frac{\beta^2\alpha^{-(\beta+1)}x_i{}^{(\beta-1)}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}[\left(\frac{x_i}{\alpha}\right)^{\beta}-1]}{[a\lambda k(1+\lambda x_i)^{-(k+1)}+\beta\alpha^{-\beta}x_i{}^{\beta-1}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}]} = 0 \quad (29)$$

$$\frac{\partial(Ln(L)f_{LW}(x;\psi))}{\partial(\beta)} = \sum_{i=1}^{n} \frac{\alpha^{-\beta}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}\beta x_i{}^{\beta-1}[\beta\left(Lnx_i-\left(\frac{x_i}{\alpha}\right)^{\beta}ln\frac{x_i}{\alpha}\right)+(1-\beta ln\alpha)]}{[a\lambda k(1+\lambda x_i)^{-(k+1)}+\beta\alpha^{-\beta}x_i{}^{\beta-1}\exp\left(\frac{-x_i}{\alpha}\right)^{\beta}]} = 0 \quad (30)$$

Solving the nonlinear equations (26) (27) (28) (29) and (30) simultaneously gives the MLE k, λ, α, β, and a in order to produce estimates of the equations describing a system of nonlinear equations, one of the numerical methods must be applied In order to obtain the estimations.

4. Conclusion

We derive new special distribution is called lomax weibull distribution (LWD). We have studied of probability density and Survival and hazard rate functions, moments, moment generating function, order statistic, generating function, which are illustrated by plotting, The estimations of k, λ, α, β, and a parameters are examined by maximum likelihood method see Figure's (5, 6, 7, 8, 9, 10, 11, 12, and 13).

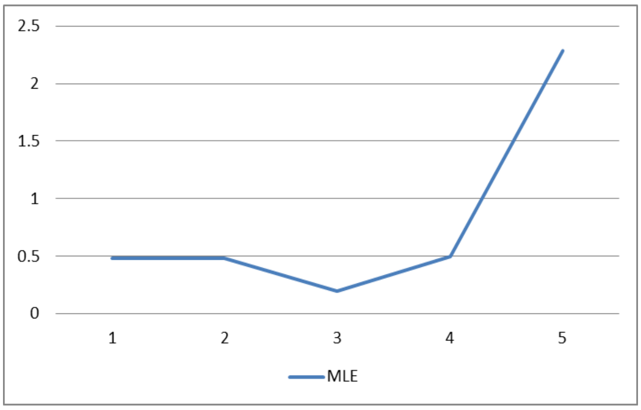

Figure 5. MLE of k

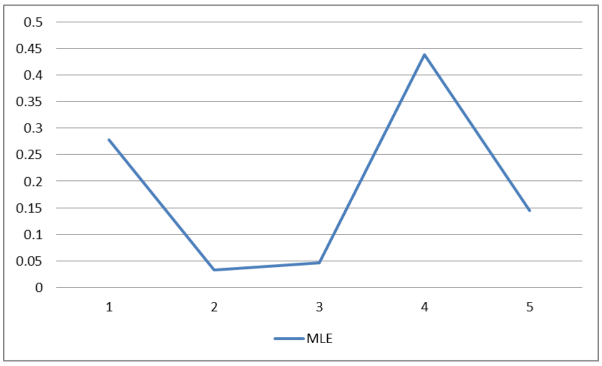

Figure 6. MLE of λ

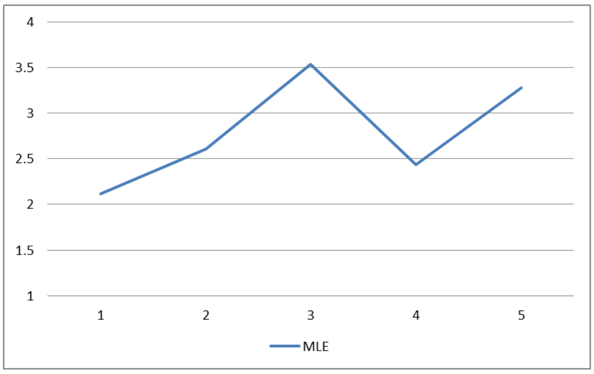

Figure 7. MLE of α

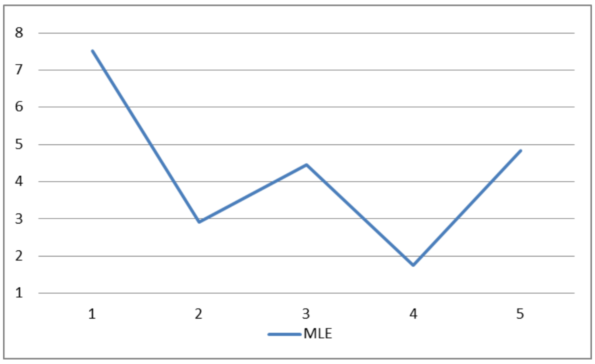

Figure 8. MLE of β

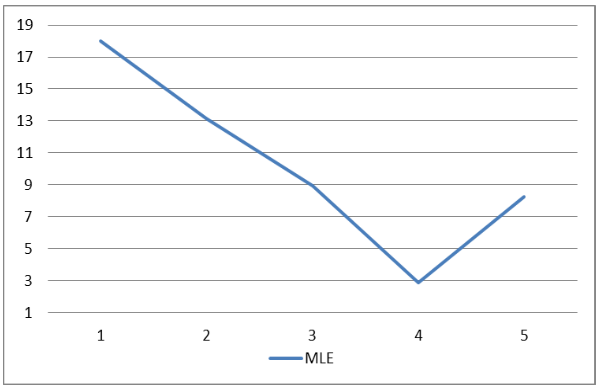

Figure 9. MLE of a

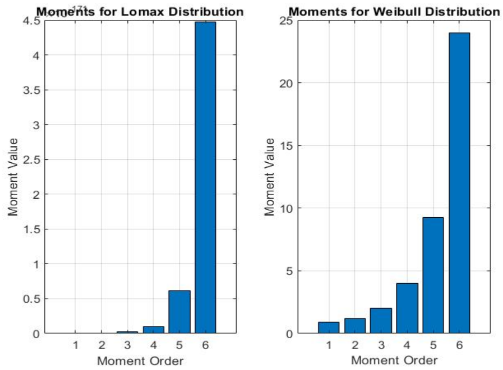

Figure 10. Moment Value of The Lomax Distribution and Weibull Distribution

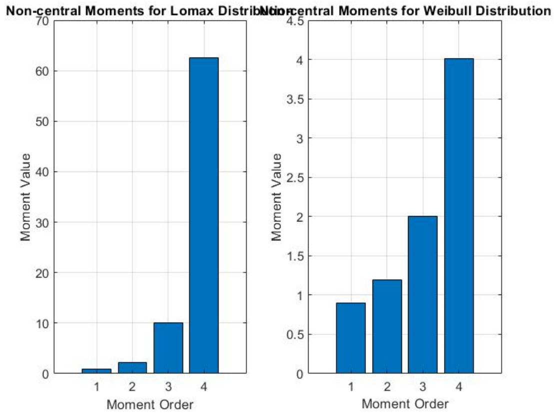

Figure 11. Non Moment Value of The Lomax Distribution and Weibull Distribution

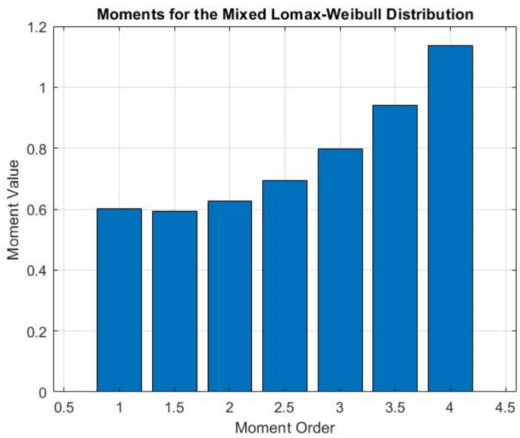

Figure 12. Moment Value of The Lomax-Weibull Distribution

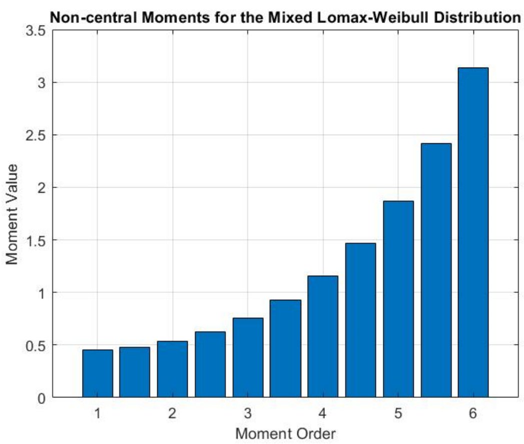

Figure 13. Non Moment Value of The Lomax-Weibull Distribution

REFERENCES

[1] M. Elgarhy, M. Shakil, and G. Kibria, "Exponentiated Weibull-Exponential Distribution with Applications," *Applications and Applied Mathematics: An International Journal (AAM)*, vol. 12, no. 2, p. 5, 2017.

[2] H. Asgharzadeh, S. Nadarajah, and F. Sharafi, "Weibull Lindley Distribution," *Revstat-Statistical Journal*, vol. 16, no. 1, pp. 87–113, 2018.

[3] G. M. Cordeiro, E. M. Ortega, and B. V. Popović, "The Gamma-Lomax Distribution," *Journal of Statistical Computation and Simulation*, vol. 85, no. 2, pp. 305–319, 2015.

[4] M. Mead, "On Five-Parameter Lomax Distribution: Properties and Applications," *Pakistan Journal of Statistics and Operation Research*, pp. 185–199, 2016.

[5] E.-H. A. Rady, W. Hassanein, and T. Elhaddad, "The Power Lomax Distribution with an Application to Bladder Cancer Data," *SpringerPlus*, vol. 5, pp. 1–22, 2016.

[6] P. E. Oguntunde, M. A. Khaleel, M. T. Ahmed, A. O. Adejumo, and O. A. Odetunmibi, "A New Generalization of the Lomax Distribution with Increasing, Decreasing, and Constant Failure Rate," *Modelling and Simulation in Engineering*, vol. 2017, no. 1, p. 6043169, 2017.

[7] M. Anwar and J. Zahoor, "The Half-Logistic Lomax Distribution for Lifetime Modeling," *Journal of Probability and Statistics*, vol. 2018, no. 1, p. 3152807, 2018.

[8] G. K. A. D. R. Abdalimam et al., "Estimation of Parameters and Survival Function for a Proposed Mixed Distribution Using Some Estimation Methods by Simulation," *Gulf Economist*, vol. 38, no. 54, 2022.

[9] U. K. Abdullahi and T. G. Ieren, "On the Inferences and Applications of Transmuted Exponential Lomax Distribution," *International Journal of Advanced Statistics and Probability*, vol. 6, no. 1, pp. 30–36, 2018.

[10] G. K. Abdulhussein and R. A. Zalan, "Estimation of the Survival Function Using the New Mixed Distribution (Power Function-Truncated Burr III) with Application," *Mathematical Statistician and Engineering Applications*, vol. 71, no. 4, pp. 4955–4970, 2022.

[11] K. S. Lomax, "Business Failures: Another Example of the Analysis of Failure Data," *Journal of the American Statistical Association*, vol. 49, no. 268, pp. 847–852, 1954.

[12] A. A. Balkema and L. De Haan, "Residual Life Time at Great Age," *The Annals of Probability*, vol. 2, no. 5, pp. 792–804, 1974.

[13] Baklizi, A. Saadati Nik, and A. Asgharzadeh, "Likelihood and Bayesian Inference in the Lomax Distribution Under Progressive Censoring," 2022.

[14] H.-C. Yeh, B. C. Arnold, and C. A. Robertson, "Pareto Processes," *Journal of Applied Probability*, vol. 25, no. 2, pp. 291–301, 1988.

[15] M. Chahkandi and M. Ganjali, "On Some Lifetime Distributions with Decreasing Failure Rate," *Computational Statistics & Data Analysis*, vol. 53, no. 12, pp. 4433–4440, 2009.

[16] A. N. Saleh, "The Effect of Extreme Values on Lomax Distribution Estimators with Simulation," *International Journal of Nonlinear Analysis and Applications*, vol. 14, no. 1, pp. 451–458, 2023.

[17] W. Weibull, "A Statistical Theory of Strength of Materials," *IVB-Handl.*, 1939.