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Article

Mixed Structure of Approximate Serre Cofibration

Aqeel Jassim Noor[1], Daher Waly Al Baydli[2]

[1,2] Departement of Mathematics, College of Education for Pure Science, University of Wasit, Wasit, Iraq

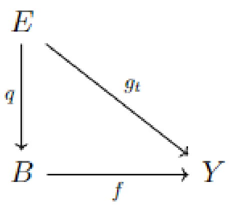
* Correspondence: aqeel.noor@uowasit.edu.iq, daheralbaydli@uowasit.edu.iq

Abstract: This paper introduces a novel concept called Mixed Approximate Serre Cofibration (MASCof), aimed at extending the structure of the traditional Serre cofibration. The main objective of this research is to define the mixed structure and prove that the majority of theorems valid for Serre cofibrations also apply to mixed approximate Serre cofibrations. Using a qualitative mathematical approach, the study demonstrates properties of MASCof and its application within CW-complex spaces. Several key theorems and propositions are presented, proving that products, pullbacks, and closed subspaces of mixed approximate Serre cofibrations retain their structural properties. The results provide insights into the homotopy extension property and mixed neighborhood deformation retracts, broadening the understanding of topological constructs.

Keywords: CW- Complex , Lowering Homotopy Property , Mixed Approximate Serre Cofibration, Homotopy Extension Property, M- Criterion.

1. Introduction

In this paper there is a description of the Mixed approximate Serre cofibration and approximate homotopy extension property. In [1,2,3] mention if $q : E \to B$ is a function, we say that q has an approximate lowering homotopy property (ALHP) with respect to X provided that given ξ is open cover of X iff given a map $f: B \to Y$ and a homotopy $g_t: E \to Y$ satisfying $f \circ q = g_0$,then there exists a homotopy $f_t: B \to Y$ with $f_0 = f$ and $f_t \circ q$ is ξ-closed g_t for all $t \in I$.



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Also [1,4] The map q is Cofibration if it has the approximate lowering homotopy property. In addition to that [1,5] assume $q: E \to B$ be a continuous function of spaces , q has the approximate lowering homotopy property (ALHP) with respect to a CW- complex space Y.

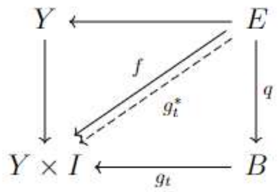


Diagram 1 (L.H.P)

If Y is the space of all CW- complex space or (all finitely triangulable spaces) , then $q: E \to B$ is called an approximate Serre cofibration [1].

2. Materials and Methods

1. Theoretical Framework
   - Overview: This section will outline the theoretical concepts and structures being studied, such as the Mixed Approximate Serre Cofibration (MASCof) and related homotopy properties. You would define the Approximate Lowering Homotopy Property (ALHP), Mixed Approximate Lowering Homotopy Property (MALHP), and explain their application to CW-complex spaces and fibrations [2].
   - Key Definitions: Include formal definitions like approximate Serre cofibrations, approximate fiber homotopy extension properties (AFHEP), and mixed approximate Serre cofibrations [3].
2. Mathematical Methods and Proofs
   - Proof Strategy: This section should describe how the theorems in your research are proven. You will outline the logical steps used, focusing on:
   - Proposition 2.1 to 2.3: Proving that every Serre cofibration is a Mixed Serre cofibration, and that product and pullback of Mixed Serre cofibrations preserve this property.
   - Homotopy Extension Property: Discuss the conditions under which a map or space possesses HEP, AFHEP, and the role of compact Hausdorff spaces.
   - Diagrammatic Proofs: Some of your theorems involve diagrams (as shown in your paper). Explain how these help visualize the homotopies and fibrations involved in proving theorems.
3. Constructive Methods
   - Step-by-Step Constructions: For certain proofs, such as the retract constructions (e.g., Theorem 3.6 and Theorem 3.7), you should detail the steps involved in creating continuous maps or retractions. This method involves constructing homotopies explicitly based on deformation retracts.
4. Application of Topological Spaces
   - Fibrations and CW-Complexes: You may describe how these topological spaces are handled in the research [4], particularly focusing on compact Hausdorff spaces and how the Serre cofibration applies in these cases. Also, include a

discussion of how CW-complexes provide a structure that allows the application of ALHP and MALHP [5].

5. M-Serre Cofibration Criteria
   - Mixed Neighborhood Deformation Retract (MNDR): This criterion is used to recognize whether a map is an M-Serre Cofibration. You should explain the derivation of the MNDR pair and how it helps identify maps that behave homologically like their quotient spaces.
6. Diagrams and Visual Aids
   - Use of Diagrams: Diagrams that accompany theorems (such as those in the file) should be referenced explicitly, explaining how they represent relationships between homotopies, fibrations, and cofibrations.

3. Results

3.1 Mixed approximate Serre cofibration

let Z be a CW-complex $f_1: E_1 \to Z$, $f_2: E_2 \to Z$ are two fiber space and $\alpha: E_2 \to E_1$ such that $f_1 \circ \alpha = f_2$, let $E_i = \{E_1, E_2\}$, $f_i = \{f_1, f_2\}$ *the* $\{E_i, f_i, Z, \alpha\}$, has Mixed approximate lowering homotopy property (MALHP) [2] with respect to a CW-space Y iff given a map $k : Z \to Y$ and a homotopy $g_t: E_1 \to Y$ satisfying $k \circ f_2 = g_0 \circ \alpha$ then there exist a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_1 = g_t$ for all $t \in I$. M-fiber space is called M approximate Serre cofibration .

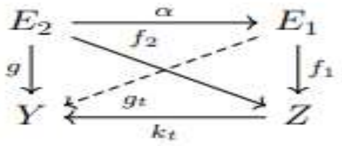


Diagram 2 (M-L.H.P)

Proposition 1.1. Every Serre cofibration is Mixed Serre cofibration.

Proof:

Let $\{E_i, f_i, Z, \alpha\}$ be a M-fiber space such that $E_1 = E_2 = E$, $\alpha =$ identity and $f = f_1 = f_2$. Let $k: Z \to Y$ and a homotopy $g_t: E_1 \to Y$ such that $k \circ f_2 = g_0 \circ \alpha$, then there exist a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_2 = g_t$ for all $t \in I$ then f_i has (MALHP) with respect to Y . Therefor f_i has M approximate Serre cofibration .

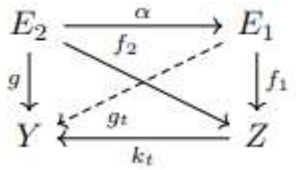


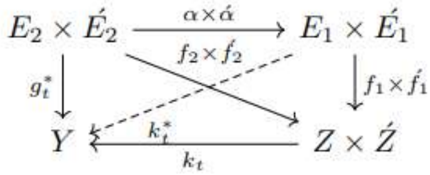
Diagram 3 (M-S cofibration)

Proposition 3.1.1. Let $f_i: E_i \to Z$ and $\acute{f_i}: \acute{E_i} \to Z$ be two M approximate Serre cofibration then $f_i \times \acute{f_i}: E_i \times \acute{E_i} \to Z \times Z^{\prime}$ is also M approximate Serre cofibration.

Proof:

Let Y be a CW-complex space , let $k^*: Z \times \acute{Z} \to Y$ be map where $k: Z \to Y$ and $\acute{k}: \acute{Z} \to Y$

Define $g_t^*: E_1 \times E_1' \to Y$ as $k^* \circ (f_2 \times f_2') = g_0^* \circ (\alpha \circ \alpha')$ such that $g_t': E_1' \to Y$ and $g_t: E_1 \to Y$,since f_i, f_i' are M approximate Serre cofibration .
Then, there exists a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_1 = g_t$ and a homotopy $k_t: Z' \to Y$ with $k_0' = k'$ and $k_t \circ f_1' = g_t$. Now, for $k_t^*: Z \times Z' \to Y$ define as $k_t^* \circ (f_1 \times f_1') = g_t^*$ and $k_0^* = k^*$. Therefore, $f_i \times f_i': E_i \times E_i' \to Z \times Z'$ is M-Serre cofibration.



Proposition 3.1.2. Let $f_i: E_i \to Z$ and $f_i': E_i' \to Z$ be two mixed approximate Serre cofibration then mixed pullback of mixed approximate Serre cofibration is also M approximate Serre cofibration [6].

Proof:

Let $k': Z' \to Y$ and $k: Z \to Y$. Define a homotopy $g_t: E_1 \to Y$ such tha $k \circ f_2 = g_0 \circ \alpha$, since f_i has M-Serre cofibration then there exists a homotopy $k_t: Z \to Y$ with $k_0 = k$ and $k_t \circ f_1 = g_t$.
Defin $g_t': E_1' \to Y$ such that $k' \circ f_2' = g_0' \circ \alpha$ and $g_t' = g_t \circ l$ then there exists a homotopy $k_t': Z' \to Y$ with $k_0' = k$ and $k_t' \circ f_1' = g_t$.Therefore $f_i': E_i' \to Z'$ has mixed approximate Serre cofibration [7].

3.2 Homotopy Extension Property

A subset $A \subset Y$ is said to have the homotopy extension property (HEP) with respect to Y , iff any partial homotopy:

$$\underline{H}: Y \times \{0\} \cup A \times I \to Z$$

Definition 3.2.1 [4] A subset $A \subset Y$ has the approximate Fiber Homotopy Extension Property (AFHEP) iff for any Serre fibration $q: E \to B$ and map $G: Y \times \{0\} \cup A \times I \to E$ such that $qG(y,t) = qG(y,0)$ where $y \in Y, 0 \le t \le 1$ there is an extension $H: Y \times I \to E$ of G such that $qG(y,t) = qG(y,0)$.

Definition 3.2.2. A subset $A_1, A_2 \subset Y$ has the mixed approximate fiber homotopy extension property (MAFHEP) iff for any M-Serre fibration $q_i : E_i \to B$ where $i = 1,2$, and map $G_1: Y \times \{0\} \cup A_1 \times I \to E_1$, $G_2: Y \times \{0\} \cup A_2 \times I \to E_2$ such that
$p_1 G_1(y_1, t) = p_1 G_1(y_1, 0)$, $p_2 G_2(y_2, t) = p_2 G_2(y_2, 0)$, where $y_1, y_2 \in Y, 0 \le t \le 1$, there is an extention
$H_1: Y \times I \to E_1$, $H_2: Y \times I \to E_2$ of G_1, G_2 such that $q_1 H_1(y_1, t) = q_1 H_1(y_1, 0)$, $q_2 H_2(y_2, t) = q_2 H_2(y_2, 0)$.

Theorem 3.2.3. If $q_i: E_i \to B$ is Mixed approximate Serre cofibration of a compact Hausdorff E_i into a Hausdorff space B, then q_i must be an imbedding.

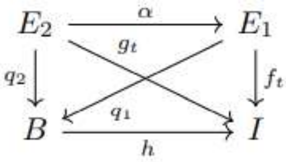
Proof: First let us prove that q_i must be injective. This will be proved by contradiction [8].
Assume that q_i were not one-one. Then, there exist two distinct points u_i and v_i in E_i with $p(u_i) = p(v_i)$. Since E_i is normal, and $\{u_i\}, \{v_i\}$ are closed subsets of E_i, there exists a continuous function

$$\lambda_1: E_1 \to I$$

And

$$\lambda_2: E_2 \to I$$

where $I=[0,1]$, with $\lambda_1(u_1)=0, \lambda_1(v_1)=1$, and $\lambda_2(u_2)=0, \lambda_2(v_2)=1$.
Define a homotopy $f_t: E_1 \to I$ and $g_t: E_2 \to I$ by $f_t(e_1)=t\lambda_1(e_1)$and $g_t(e_2)=t\lambda_2(e_2)$, $e_1 \in E_1, e_2 \in E_2, t \in I$ and define $h: B \to I$ by $h(b)=0$ for all $b \in B$



Then, we have $hq_1=f_0$ and $hq_2=g_0$.
But $q_1: E_1 \to B$ and $q_2: E_2 \to B$ are a M-Serre cofibration, hence there exists a homotopy $h_t: B \to I$ such that $h_0=h$, and $h_t q_1 = f_t$,
$h_t q_2 = g_t$ for all $t \in I$. Now

$$\lambda_1(u_1)=f_1(u_1)=h_1(q_1(u_1))$$
$$\lambda_1(v_1)=f_1(v_1)=h_1(q_1(v_1))$$

And

$$\lambda_2(u_2)=g_1(u_2)=h_1\big(q_2(u_2)\big)$$
$$\lambda_2(v_2)=g_1(v_2)=h_1\big(q_2(v_2)\big)$$

Since $q_1(u_1)=q_1(v_1)$, therefore $\lambda_1(u_1)=\lambda_1(v_1)$ and $q_2(u_2)=q_2(v_2)$ therefore $\lambda_2(u_2)=\lambda_2(v_2)$ But

$$\lambda_1(u_1)=0 \neq 1=\lambda_1(v_1)$$

And

$$\lambda_2(u_2)=0 \neq 1=\lambda_2(v_2)$$

Therefore q_i must be injective. Since E_i is compact Hausdorff and B is Hausdorff, hence q_i must be an imbedding.
Hence $q_1: E_1 \to B$ and $q_2: E_2 \to B$ are a mixed approximate Serre cofibration, when $E_i \subset B$ has the HEP

with respect to all CW-complex spaces.

Theorem 3.2.4. Let A_i be a closed subspace of topological space Y_i. Where $i=1,2$. Then (Y_i, A_i) is a Mixed approximate Serre cofibered pair if and only if there exists :

1) A neighborhood U_i of $A_i$which is deformable in Y_i to A_i $relA_i$ (there exists a homotopy $H_1: U_1 \times I \to Y_1$ such that $H_1(y_1,0)=y_1$, $H_1(a_1,t)=a_1$and $H_2: U_2 \times I \to Y_2$ such that
$H_2(y_2,0)=y_2, H_2(a_2,t)=a_2$and $H_1(y_1,0) \in A_1, H_2(y_2,0) \in A_2$ for all $y_1 \in U_1 \wedge y_2 \in U_2$, $a_1 \in A_1 \wedge a_2 \in A_2$, $t \in I$.
2) A continuous function $\varphi_i: Y_i \to I$ such that $A_1=\varphi_1^{-1}(0)$, $\varphi_1(y_1)=1$ and $A_2=\varphi_2^{-1}(0)$, $\varphi_2(y_2)=1$ for all $y_1 \in Y_1-U_1$, $y_2 \in Y_2-U_2$.

Proof: Suppose that (Y_i, A_i) is a M-Serre cofibered pair. Then there exists a retraction

$$r_i: Y_i \times I \to (Y_i \times 0) \cup (A_i \times I)$$

Where $i=1,2$, and U_i, H_i and φ_i may be chosen as follows:

$$U_i=\{pr_i r_i(y_i,1) \in A_i\}$$

$$H_i=pr_i r_i | U_i \times I$$

$$\varphi_i(y_i)=sup\ t \in I\ |t-pr_2 r_i(y_i,t)|$$

pr_1 and pr_2 denoting projections on Y_i and I, respectively.
Conversely, suppose that U_i, H_i and φ_i are given and satisfy the conditions of the theorem. Since A_i is closed it suffices to prove the existence of a retraction.

$$r_i: Y_i \times I \to (Y_i \times 0) \cup (A_i \times I)$$

The required retraction may be constructed as follows:

- If $\varphi_i(y_i) = 1$, *let* $r_i(y_i, t) = (y_i, 0)$.
- If $\frac{1}{2} \leq \varphi_i(y_i) < 1$, let $r_i(y_i, t) = \{(H_1(y_1), 2(1 - \varphi_1(y_1))t), 0).(H_2(y_2), 2(1 - \varphi_2(y_2))t), 0)$.
- If $0 < \varphi_i(y_i) \leq \frac{1}{2}$ and $0 \leq t \leq 2\varphi_i(y_i)$, let $r_i(y_i, t) = \{(H_1(y_1, \frac{t}{(2\varphi_1(y_1)}))$,0).$(H_2(y_2, \frac{t}{(2\varphi_2(y_2)}))$,0).
- If $0 < \varphi_i(y_i) \leq \frac{1}{2}$ and $2\varphi_i(y_i) \leq t \leq 1$, let $r_i(y_i, t) = \{(H_1(y_1, 1), t - 2\varphi_1(y_1)).(H_2(y_2, 1), t - 2\varphi_2(y_2))$.
- If $\varphi_i(y_i) = 0$, let $r_i(y_i, t) = (y_i, t)$.

Lemma 3.2.5. If (Y_i, A_i) is mixed approximate Serre cofibered pair, where $i = 1,2$, then $(Y_i \times 0) \cup (A_i \times I)$ is a strong deformation retract of $Y_i \times I$.

Proof: Let $\eta_i: (Y_i \times 0) \cup (A_i \times I) \subset Y_i \times I$ be the inclusion map, and let

$$r_i : Y_i \times I \to (Y_i \times 0) \cup (A_i \times I)$$

be a retraction. A homotopy

$$D_1: \eta_1 r_1 \simeq 1_{Y_1 \times I} \quad rel(Y_1 \times 0) \cup (A_1 \times I)$$

And

$$D_2: \eta_2 r_2 \simeq 1_{Y_2 \times I} \quad rel(Y_2 \times 0) \cup (A_2 \times I)$$

Is given by

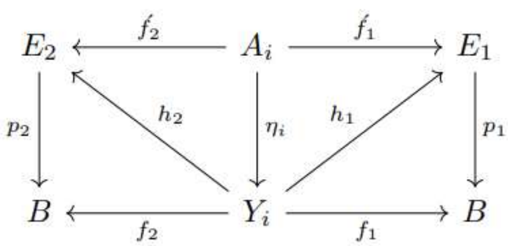
$$D_i(y_i, t, \acute{t}) = (pr_j r_i(y_i, (1 - \acute{t})t), (1 - \acute{t}) pr_l r_i(y_2, t) + \acute{t}t)$$

Where $j = 1,2$, $l = 3$.

$$D_1(y_1, t, \acute{t}) = (pr_1 r_1(y_1, (1 - \acute{t})t), (1 - \acute{t}) pr_3 r_1(y_1, t) + \acute{t}t).$$

$$D_2(y_2, t, \acute{t}) = (pr_2 r_2(y_2, (1 - \acute{t})t), (1 - \acute{t}) pr_3 r_2(y_2, t) + \acute{t}t).$$

Theorem 3.2.6. Suppose that $p_i: E_i \to B$ is M-fibration, that A_i is a strong deformation retract of Y_i, and that there exists a map $\varphi_i: Y_i \to I$ such that $A_1 = \varphi_1^{-1}(0)$and $A_2 = \varphi_2^{-1}(0)$.Then any commutative diagram (Pacheco, 2024).



may be filled in with a map $h_i: Y_i \to E_i$ such that $p_1 h_1 = f_1$, $p_2 h_2 = f_2$ and $h_1 \eta_1 = \acute{f}_1$, $h_2 \eta_2 = \acute{f}_2$. h_i is unique up to homotopy $relA_i$.

Proof: By hypothesis there exists a retraction $r_1: Y_1 \to A_1$,$r_2: Y_2 \to A_2$and a homotopy

$$D_1: \eta_1 r_1 \simeq 1_{Y_1} \quad relA_1 ,$$

$$D_2: \eta_2 r_2 \simeq 1_{Y_2} \quad relA_2.$$

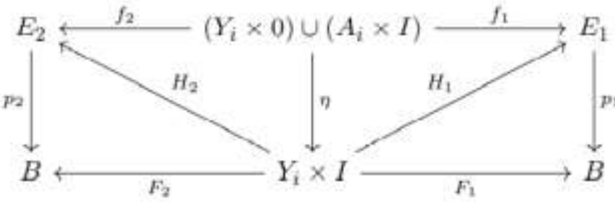
If $h_1: Y_1 \to E_1$ and $h_2: Y_2 \to E_2$ such that $h_i \eta_i = f_i'$, then $h_i \simeq h_i \eta_i r_i = f_i'\, relA$ which proves the last assertion of the theorem. Define $\underline{D_i}: Y_i \times I \to Y_i$ by

$$\underline{D_1}(y_1, t) = \{D_1(y_1, \frac{t}{(\varphi_1(y_1))} \quad t < \varphi_1(y_1) \quad D_1(y_1, 1) \qquad t \geq \varphi_1(y_1)$$

$$\underline{D_2}(y_2, t) = \{D_2(y_2, \frac{t}{(\varphi_2(y_2))} \quad t < \varphi_2(y_2) \quad D_2(y_2, 1) \qquad t \geq \varphi_2(y_2)$$

D_i is easily shown to be continuous. Because p_i is a fibration there exists a homotopy $\underline{F_1}: Y_1 \times I \to E_1$, $\underline{F_2}: Y_2 \times I \to E_2$ such that $p_1 \underline{F_1} = f_1 \underline{D_1}$, $p_1 \underline{F_1} = f_1 \underline{D_1}$ and $\underline{F_1}(y_1, 0) = f_1' r_1(y_1)$, $\underline{F_2}(y_2, 0) = f_2' r_2(y_2)$ for each $y_1 \in Y_1, y_2 \in Y_2$. h_i is given by $h_i(y_i) = \underline{F_i}(y_i, \varphi_i(y_i))$, where $i = 1,2$.

Theorem 3.2.7. Suppose that $p_i: E_i \to B$ is a mixed approximate fibration, that (Y_i, A_i) is a mixed approximate Serre cofibered pair, and that A_i is closed. Then any commutative diagram



may be filled in with a homotopy $\underline{F}: Y_i \times I \to E_i$ such that $p_1 \underline{F_1} = F_1$, $p_2 \underline{F_2} = F_2$ and $\underline{F_i}|(Y_i \times 0) \cup (A_i \times I) = f_i$.

Proof: According to the lemma(4.5), and by theorem(4.4) there exists a Function $\psi_1: Y_1 \to I$, $\psi_2: Y_2 \to I$ such that $A_1 = \psi_1^{-1}(0), A_2 = \psi_2^{-1}(0)$.
Define $\varphi_1: Y_1 \times I \to I$ and $\varphi_2: Y_2 \times I \to I$ by $\varphi_1(y_1, t) = t\psi_1(y_1)$and $\varphi_2(y_2, t) = t\psi_2(y_2)$. Then $(Y_1 \times 0) \cup (A_1 \times I) = \varphi_1^{-1}(0)$and $(Y_2 \times 0) \cup (A_2 \times I) = \varphi_2^{-1}(0)$ and the theorem follows from theorem(4.6) The condition that A_i be closed is not very restrictive. For instance, A will always be closed if Y_i is Hausdorff. Not all M-Serre cofibration are closed. The most trivial example of a non-closed M-Serre cofibration is the pair $(Y_1, a_1), (Y_2, a_2)$ where Y_i is the two-point space a_i, b_i with the trivial topology

3.3. A mixed criterion for a map to be a mixed Approximate Serre Cofibration

In this section the M-criterion that allows us to recognize M-Serre Cofibration when we see them [10]. We shall often consider pair (Y_i, A_i) consisting of a space Y_i and a subspace A_i. M-Serre Cofibration pairs will be those pairs that "behave homologically" just like the associated quotient spaces $\frac{Y_i}{A_i}$.

Definition 3.3.1. A pair (Y_i, A_i) is an Mixed Neighborhood Deformation Retract pair (MNDR-pair) if there is a map $u: Y_1 \to I$, $v: Y_2 \to I$ such that $u^{-1}(0) = A_1$, $v^{-1}(0) = A_2$ and a homotopy $h: Y_1 \times I \to Y_1$, $k: Y_2 \times I \to Y_2$ such that $h_0 = id$, $h(a_1, t) = a_1$ and $k_0 = id$, $k(a_2, t) = a_2$ for $a_1 \in A_1, a_2 \in A_2$ and $t \in I$, and $h(y_1, 1) \in A_1$ if $u(y_1) < 1$, and $k(y_2, 1) \in A_2$ if $v(y_2) < 1$. (Y_i, A_i) is a MDR-pair if $u(y_1) < 1$, $v(y_2) < 1$ for all $y_1 \in Y_1, y_2 \in Y_2$, in which case A_i is a deformation retract of Y_i where $i = 1,2$.

Lemma 3.3.2. If (h_i, u_i) and (k_i, v_i) represent (Y_i, A_i) and (Z_i, B_i) as (MNDR-pairs), then (l_i, w_i) represents the (product pair) $(Y_i \times Z_i, Y_i \times B_i \cup A_i \times Z_i)$ as an MNDR-pair, where $w_i(y_i, z_i) = min\ (u_i(y_i), v_i(z_i))$ and

$$l_i(y_i, z_i, t) = \{\left(h_i(y_i, t), k_i\left(z_i, \frac{tu_i(y_i)}{v_i(z_i)}\right)\right) \quad if\ v_i(z_i)$$

$$\geq u_i(y_i)\ \ (h_i\left(y_i, \frac{tv_i(z_i)}{u_i(y_i))}, k_i(z_i, t)\right) \quad if\ u_i(y_i) \geq v_i(z_i)$$

If (Y_i, A_i) or (Z_i, B_i) is a DR-pair, then so is $(Y_i \times Z_i, Y_i \times B_i \cup A_i \times Z_i)$.

Proof: If $v_i(z_i) = 0$ and $v_i(z_i) \geq u_i(y_i)$, then $u_i(y_i) = 0$ and both (Z_i, B_i) and (Y_i, A_i) , therefore we can and must understand $l_i(y_i, z_i, t)$ to be (y_i, z_i). It is easy to check from this and the symmetric observation that l_i is a well defined continuous homotopy as desired [11].

Theorem 3.3.3. Let A_i be a closed subspace of Y_i, where $i = 1,2$. Then the following are equivalent:

- (Y_i, A_i) is an MNDR-pair.
- $(Y_i \times I, Y_i \times \{0\} \cup A_i \times I)$is a MDR-pair.
- $Y_i \times \{0\} \cup A_i \times I$ is a M-retract of $Y_i \times I$.
- The inclusion $\eta_i: A_i \to Y_i$ is a M-Serre cofibration.

Proof: The lemma gives that (a) implies (b), (b) trivially implies (c), and we have already seen that (c) and (d) are equivalent. Assume given a retraction $r_i : Y_i \times I \to Y_i \times \{0\} \cup A_i \times I$.
Let $pr_i: Y_i \times I \to Y_i$ and $pr_2: Y_i \times I \to I$ be the projections and define $u: Y_1 \to I$ by
$u(y_1)$ =$sup\ sup\ \{t - pr_2\ r_1(y_1, t) | t \in I\}$,
and $v: Y_2 \to I$ by

$$v(y_2) = sup\ sup\ \{t - pr_2\ r_2(y_2, t) | t \in I\}$$

$h: Y_1 \times I \to Y_1$ by

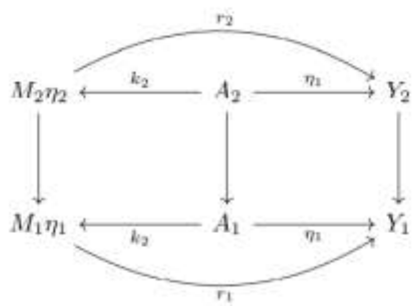
$$h(y_1, t) = pr_1 r_1(y_1, t)$$

and $k: Y_2 \times I \to Y_2$ by

$$k(y_2, t) = pr_2 r_2(y_2, t)$$

Then $(h, u), (k, v)$ represents (Y_i, A_i) as an MNDR-pair. Here $u^{-1}(0) = A_1$ since $u(y_1) = 0$ and $v^{-1}(0) = A_2$ since $v(y_2) = 0$ implies that $r_i(y_i, t) \in A_i \times I$ for $t > 0$ and thus also for $t = 0$ since $A_i \times I$ is closed in $Y_i \times I$,where $i = 1,2$.

Example 3.3.4. Let $\eta_1: A_1 \to Y_1$and $\eta_2: A_2 \to Y_2$ be a M-Serre cofibration, where $i = 1,2$. We then have the commutative diagram



Where $k_1(a_1) = (a_1, 1)$ and $k_2(a_2) = (a_2, 1)$ where $M_i\eta_i \equiv Y_i \cup_\eta (A_i \times I)$. The obviou homotopy inverse $l_i: Y_i \to M_i\eta_i$ has $l_i(y_i) = (y_i, 0)$ and is thus very far from being a map A_i. The proposition ensures that l_i is homotopic to a map under A_i that is homotopy inverse to r_i under A_i

3. Conclusion

We mention the most important result that reached through our study mixed structure Serre cofibration as following:

- Every approximate Serre cofibration is Mixed approximate Serre cofibration.
- product Two Mixed approximate Serre cofibration is also Mixed approximate Serre cofibration [12].
- M-pullback of Mixed approximate Serre cofibration is also Mixed approximate Serre cofibration.

Let A_i be a closed subspace of Y_i, Then equivalent: The inclusion $\eta_i: A_i \to Y_i$ is a mixed approximate Serre cofibration [13].

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