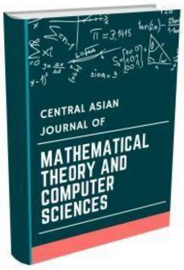

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Article

Using Optimal Homotopy Asymptotic Method for Solving Differential Equations

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Abstract: In this study, we address the challenge of solving both linear and nonlinear differential equations by employing the Optimal Homotopy Asymptotic Method (OHAM), a novel semi-analytic approximation technique. Existing methods like the Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), and Homotopy Analysis Method (HAM) often rely on free parameters, which can limit their effectiveness. OHAM, however, operates without this dependence, achieving higher accuracy with fewer approximations. Through comparative analysis, we demonstrate that OHAM performs consistently well over large domains and allows for easy adjustment of the convergence domain. These findings suggest that OHAM provides a more efficient and accurate alternative for solving complex differential equations.

Keywords: Optimal Homotopy Asymptotic Method, OHAM, Nonlinear differential equations, Semi-analytic methods, Convergence analysis

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1. Introduction

IVP and BVP "exact solutions are extremely rare, and they become even rarer when the equations are nonlinear. Differential equations with variable boundary conditions are the most effective means of describing most natural processes. Both analytical and numerical approaches have played important roles in the solution of scientific and technical problems. These techniques don't always produce exact solutions for nonlinear problems. since developing alternative strategies to tackle the challenging non-linear situations is vital. Collocation methods [1,2], finite element methods [3-5], finite difference methods [6], radial basis function collocation method [7–17], and other well-known numerical techniques are examples.

When it comes to solving nonlinear problems in simple domains, numerical methods [18] have an advantage over analytical methods in that they are less expensive in complicated domains. The development of diverse analytical techniques to guarantee time and cost efficiency in issue solutions has advanced significantly.Liao used homotopy to create the Homotopy Analysis Method (HAM) [19–25]. In HAM, the author splits a nonlinear problem into numerous linear subproblems by using the fundamental topological idea of homotopy. A generalized Taylor series with regard to an embedding parameter is where HAM originates [26].

Numerous varieties of nonlinear mathematical models are solved by HAM. Whether or whether tiny parameters are present in the problems being studied has no bearing on the validity of the HAM.It allows you a lot of freedom when selecting auxiliary linear operators. "It has been successfully applied to many nonlinear problems such as nonlinear oscillations, boundary layer flows, heat transfer, viscous flows in porous media, viscous flows of Oldroyd 6-constant fluids, magneto hydrodynamic flows of nonnewtonian fluids, nonlinear water waves, Thomas-Fermi equation, Lane-Emden equation, and so on." [18].The Optimization Homotopy Asymptotic Method (OHAM) was introduced by Vasile Marinca et al. and [27–31] N. Herisanu et al. for the approximate solution of "nonlinear problems of thin film flow of a 4th degree fluid down a vertical cylinder" and for the behavior analysis of "nonlinear mechanical vibration of electrical machine."

The same authors employed this technique to solve "nonlinear equations arising in heat transfer" as well as "nonlinear equations arising in the steady state flow of a fourth-grade fluid past a porous plate." OHAM has recently been applied to the "determination of periodic solutions for the motion of a particle on a rotating parabola" by Marinca et al. OHAM has been used more recently in the "solution of multipoint boundary value problems" by Javed et al. [32]. [33] describes the application of OHAM to a Jefferson-Hamel flow, where fluid flows between two stiff plane walls at an angle of 2α. OHAM has been used to study the "steady incompressible mixed convection flow past vertical flat plate" problem in [34]".

2. Materials and Methods

This study employs the Optimal Homotopy Asymptotic Method (OHAM) to solve both linear and nonlinear differential equations. The main steps of the method are as follows:

1. Defining the Differential Equation
   The process begins by selecting the differential equation to be solved, whether linear or nonlinear.
2. Constructing the Homotopy Equation:
   Using OHAM, a homotopy equation is created, linking an initial rough approximation to the desired solution. An "embedding parameter" is introduced to control the transition between these solutions.
3. Expanding the Solution Series:
   The solution to the differential equation is expanded into a Taylor series, where each stage of the expansion provides a solution closer to the exact value.
4. Calculating the Residual:
   The residual represents the difference between the calculated solution and the exact solution. This residual is calculated at each stage to assess how far the solution is from the true value.
5. Minimizing the Residual:
   The Least Squares method is used to minimize the residual, ensuring the obtained solution is as optimal as possible. This step adjusts certain constants to achieve a more accurate result.
6. Iterative Approach:
   The process is repeated iteratively until the solution reaches the desired accuracy, with each iteration improving the approximation.

Materials

a. Mathematical Models:
   The study works with differential equations as the foundational models, both linear and nonlinear. A third-order boundary value problem is used as an example of OHAM’s application.
b. Software Tools:

The calculations and simulations in this study can be performed using software like MATLAB or Mathematica, which are helpful for symbolic and numerical computations.

c. Comparison Methods:
The results obtained from OHAM are compared with other methods like the Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM), and Homotopy Analysis Method (HAM). This comparison helps evaluate the accuracy and effectiveness of OHAM in different situations.

3. Results and Discussion

2. Basic Concepts

2.1 Homotopy [35]:

"Geometric homotopy studies the general characteristics of space and curves. It can be conceptualized as the study of characteristics like twisting and stretching that are unaffected by ongoing deformations. When points in one figure match exactly to points in another, a transformation takes place. The term "homotopical equivalent" refers to a set of figures that can be turned into each other. Consequently, "if one continuous function can be continuously deformed into the other, two continuous functions from one topological space to another are called homotopic." Between the two functions, such a distortion is referred to as a homotopy. That is, $H(\eta,\varepsilon):\Omega\times[0,1]\rightarrow R;\eta\in\Omega,\varepsilon\in[0,1]$, where H represents the original function f's deformation".

$$\mathcal{H}[\eta,\varepsilon] = (1-\varepsilon)f_1(\tau) + \varepsilon f_2(\tau) \tag{2.1}$$

would be a homotopy between the functions $f_1(\tau)$ and $f_2(\tau)$.Here, ε is called the embedding parameter and if $\varepsilon = 0, \mathcal{H}(\tau,0) = f_1(\tau)$ and if $\varepsilon = 1, \mathcal{H}(\tau,1) = f_2(\tau)$. Thus, as ε changes from 0 to 1, $f_1(\tau)$ is gradually transformed into $f_2(\tau)$ ".

2.2 Residual [5].

"The difference between the estimated and true values is known as the residual. It is calculated for a differential equation by changing the original differential equation with the estimated solution".

2.3 Fundamental mathematical theory of OHAM [27-31]

"Consider the differential equation below to demonstrate the fundamental principle" of OHAM:

$$\mathcal{L}(w(\eta)) + g(\eta) + \mathcal{N}(w(\eta)) = 0 \tag{2.2}$$

$$\mathcal{B}\left(w, \frac{dw}{d\eta}\right) = 0 \tag{2.3}$$

where w(η) is a necessary function, g(η) is a given function, and L, N, and B are boundary, nonlinear, and linear operators, respectively. η stands for independent variable. The following generic deformation equation can be created by using OHAM on the aforementioned problem :

$$(1-\varepsilon)[\mathcal{L}(\mathcal{H}(\eta,\varepsilon) + g(\eta)] = h(\varepsilon)[\mathcal{L}(\mathcal{H}(\eta,\varepsilon) + g(\eta) + \mathcal{N}(\mathcal{H}(\eta,\varepsilon))] \tag{2.4}$$

$$\mathcal{B}\left(\mathcal{H}(\eta,\varepsilon), \frac{\partial\mathcal{H}(\eta,\varepsilon)}{\partial\eta}\right) = 0 \tag{2.5}$$

where $0 \leq \varepsilon \leq 1$ is an embedding parameter, $h(\varepsilon)$ is a nonzero auxiliary function for $\varepsilon \neq 0$ and $h(0) = 0, \mathcal{H}(\eta,\varepsilon)$ is an unknown function .

It is clear that when $\varepsilon = 0$ and $\varepsilon = 1$ it holds $\mathcal{H}(\eta, 0) = w_0(\eta)$ and $\mathcal{H}(\eta, 1) = w(\eta)$ respectively.

So, as ε changes from 0 to 1, the solution $\mathcal{H}(\eta, \varepsilon)$ changes from $w_0(\eta)$ to the solution $w(\eta)$, where $w_0(\eta)$ is obtained from Eq(2.4) for $\varepsilon = 0$:

$$\mathcal{L}(w_0(\eta)) + g(\eta) = 0,\ \mathcal{B}\left(w_0, \frac{dw_0}{d\eta} = 0\right) \tag{2.6}$$

For proper applications K_i are finite, say, $i = 1,2,3, \dots, m$. We present the auxiliary function $h(\varepsilon)$ to be of the type :

$$h(\varepsilon) = \varepsilon K_1 + \varepsilon^2 K_2 + \varepsilon^3 K_3 + \cdots + \varepsilon^m K_m, \tag{2.7}$$

where $K_1, K_2, \dots, K_m$ are constants.
For solution, expanding $\mathcal{H}(\eta, \varepsilon)$ in Taylor's series about ε, we get:

$$\mathcal{H}(\eta, \varepsilon) = w_0(\eta) + \sum_{n=1}^{\infty} w_k(\eta, K_1, K_2, \dots, K_n)\varepsilon^n \tag{2.8}$$

Once we replace the value of (2.8) in equation (2.4) and calculate the coefficient of similar powers of ε, we obtain: Equation (2.6) provides the zeroth order problem, and equations (2.7)–(2.12) provide the first and second order problems, respectively :

$$\mathcal{L}(w_1(\eta)) + g(\eta) = K_1 \mathcal{N}_0(w_0(\eta)) \tag{2.9}$$

$$\mathcal{B}\left(w_1, \frac{dw_1}{d\eta}\right) = 0 \qquad (2.10)$$

$$\mathcal{L}(w_2(\eta)) - \mathcal{L}(w_1(\eta)) = K_2 \mathcal{N}_0(w_0(\eta)) + \qquad (2.10)$$

$$K_1[\mathcal{L}(w_1(\eta)) + \mathcal{N}_1(w_0(\eta), w_1(\eta))], \qquad (2.11)$$

$$\mathcal{B}\left(w_2, \frac{dw_2}{d\eta}\right) = 0 \qquad (2.12)$$

In general, we can write above equations by the following form:

$$\mathcal{L}(w_n(\eta)) - \mathcal{L}(w_{n-1}(\eta)) = K_n \mathcal{N}_0(w_0(\eta)) +$$
$$\sum_{i=1}^{n-1} K_i[\mathcal{L}(w_{n-i}(\eta)) + \mathcal{N}_{n-i}(w_0(\eta), w_1(\eta), \dots, w_{n-1}(\eta))]$$
$$\mathcal{B}\left(w_n, \frac{dw_n}{d\eta}\right) = 0$$

where $n = 2,3, \dots,$
In the above equation $\mathcal{N}_m(w_0(\eta), w_1(\eta), \dots, w_{n-1}(\eta))$ is the coefficient of ε^m in the expansion of $\mathcal{N}(\mathcal{H}(\eta, \varepsilon))$:

$$\mathcal{N}\big(\mathcal{H}(\eta, \varepsilon, K_i)\big) = \mathcal{N}_0(w_0(\eta)) + \sum_{m=1}^{\infty} \mathcal{N}_m(w_0, w_1, \dots, w_m)\varepsilon^m \tag{2.15}$$

Here, convergence of the series (2.8) depends upon the constants $K_i, i = 1,2,3, \ldots$.

When $\varepsilon = 1$ the equation (2.8) transforms to:

$$\tilde{w}(\eta, K_1, K_2, \ldots) = w_0(\eta) + \sum_{i=1}^{\infty} \tilde{w}_i(\eta, K_1, K_2, \ldots, K_m) \tag{2.16}$$

In actual application i is finite. Hence the equation (2.10) reduce to :

$$\tilde{w}(\eta, K_1, K_2, \ldots, K_m) = w_0(\eta) + \sum_{i=1}^{m} w_i(\eta, K_1, K_2, \ldots, K_m) \tag{2.17}$$

Substituting equation (2.17) into equation (2.4), then we get the following residual :

$$R(\eta, K_1, K_2, \ldots, K_m) = \mathcal{L}\big(\tilde{w}(\eta, K_1, K_2, \ldots, K_m)\big) + g(\eta) + \mathcal{N}\big(\tilde{w}(\eta, K_1, K_2, \ldots, K_m)\big) \tag{2.18}$$

If $R = 0$, then $\tilde{w}$ yields the exact solution. Generally, it doesn't happen, especially in nonlinear problems.

For the determinations constants of $K_i, i = 1,2, \ldots, m$, We select a and b in a way that results in the optimal K_i s values for the desired problem's convergent solution. The Ritz Method, Galerkin's Method, and Collocation Method are a few techniques to determine the ideal values of $K_i, i = 1,2,3, \ldots$. Here, we apply the Method of Least Squares as under :

$$S(\eta, K_1, K_2, \ldots, K_m) = \int_a^b R^2(\eta, K_1, K_2, \ldots, K_m)dx \tag{2.19}$$

where R is the residual, $R = \mathcal{L}(\tilde{w}) + g(\eta) + \mathcal{N}(\tilde{w})$ and

$$\frac{\partial S}{\partial K_1} = \frac{\partial S}{\partial K_2} = \cdots = \frac{\partial S}{\partial K_m} = 0 \tag{2.20}$$

where a and b are properly chosen numbers to locate the desired $K_i (i = 1,2, \ldots, m)$. With these constants known, the approximate solution (of order m) is well-determined .

3. Application

Example 3.1: To see the effectiveness of the above basic idea of OHAM, we are presenting an example of third order boundary value problem :

$$w'''(\eta) - \eta w(\eta) = (\eta^3 - 2\eta^2 - 5\eta - 3)e^{\eta} \tag{3.1}$$

with boundary conditions

$$w(0) = 0, w'(0) = 1, w'(1) = -e. \tag{3.2}$$

Exact solution of the problem is

$$w(\eta) = \eta(1-\eta)e^{\eta} \tag{3.3}$$

According to OHAM formulation given in equation (2.4) for the given problem (3.1), we will apply OHAM to get the following BVPs generated by OHAM procedure :

Zeroth order boundary value problem is given by :

$$w_0'''(\eta) = 0;\ w_0(0) = 0, w_0'(0) = 1, w_0'(1) = -e \tag{3.4}$$

The solution of the zeroth order deformation is :

$$w_0(\eta) = \frac{1}{2}(2\eta - \eta^2 - e\eta^2) \tag{3.5}$$

The 1st order deformation problem is given as :

$$w_1'''(\eta) = (1 + K_1)w_0'''(\eta) - \eta C_1 w_0(\eta) + 3e^{\eta}K_1$$
$$w_1(0) = w_1'(0) = w_1'(1) = 0$$

1st order solution is:

$$\begin{aligned} w_1(\eta) = {} & -\left(\frac{K_1}{240}\right)(17280 - 17280e\eta + 6960\eta + 4\eta^5 + 10320e^{\eta}\eta \\ & -3487\eta^2 + 1803e\eta^2 - 2640e^{\eta}\eta^2 + 240e^{\eta}\eta^3 - \eta^6 - e\eta^6) \end{aligned} \tag{3.6}$$

2nd order deformation problem is given as:

$$w_2'''(\eta) = (1 + K_1)w_1'''(\eta) + K_2 w_0'''(\eta) - \eta K_1 w_1(\eta) + 3e^{\eta}K_2$$
$$w_2(0) = w_2'(0) = w_2'(1) = 0$$

The 2nd order solution can be observed as:

$$\begin{aligned} w_2(\eta) = {} & \frac{1}{1209600}\Big(-5040K_1(17280 + 6960\eta - 3487\eta^2 + 4\eta^5 - \eta^6 \\ & +240e^{\eta}(-72 + 43\eta - 11\eta^2 + \eta^3) - e\eta^2(-1803 + \eta^4)\big) \\ & +K_1^2(-2206310400 - 985824000\eta + 484667297\eta^2 \\ & +1209600e^{\eta}(1824 - 1009\eta + 225\eta^2 - 24\eta^3 + \eta^4) \\ & -7e\eta^2 33730609 - 11538\eta^4 + \eta^8) \\ & -5040K_2(17280 + 6960\eta - 3487\eta^2 + 4\eta^5 - \eta^6 + 240e^{\eta} \\ & (-72 + 43\eta - 11\eta^2 + \eta^3) - e\eta^2(-1803 + \eta^4))\Big) \end{aligned}$$

Now we can see the solution as $w(\eta) = w_0(\eta) + w_1(\eta) + w_2(\eta) + \cdots$

$$w(\eta) = \frac{1}{1209600}\Big(-10080K_1(17280 + 6960\eta - 3487\eta^2 + 4\eta^5 - \eta^6 + 240e^{\eta}(-72 + 43\eta - 11\eta^2 + \eta^3) - e\eta^2(-1803 + \eta^4)) + K_1^2(-2206310400 - 985824000\eta + 484667297\eta^2 + 1209600e^{\eta}(1824 - 1009\eta + 225\eta^2 - 24\eta^3 + \eta^4) - 7e\eta^2(33730609 - 11538\eta^4 + \eta^8)) - 5040(120\eta(-2 + \eta + e\eta) - K_2(17280 + 6960\eta - 3487\eta^2 + 4\eta^5 - \eta^6 + 240e^{\eta}(-72 + 43\eta - 11\eta^2 + \eta^3) - e\eta^2(-1803 + \eta^4)))\Big)$$

The residual of the example problem is given as :

$$R = (1/2)\big((-2 + \eta)\eta^2 + e\eta^3 + e^{\eta}(6 + 10\eta + 4\eta^2 - 2\eta^3)\big) + (K_1/120)(-e\eta^3(-1923 + \eta^4) + 240e^{\eta}(3 - 67\eta + 45\eta^2 - 12\eta^3 + \eta^4) + \eta(17280 + 6720\eta - 3367\eta^2 + 4\eta^5 - \eta^6)) - (K_1^2/1209600)(1209600e^{\eta}(-3 + 1891\eta - 1054\eta^2 + 237\eta^3 - 25\eta^4 + \eta^5) - 1019692800\eta + 501636977\eta^2 + 3628800\eta^4 + 544320\eta^5 - 136374\eta^6 + 40\eta^9 - 7\eta^{10})) + (K_2/240)(-e\eta^3(-1923 + \eta^4) + 240e^{\eta}(3 - 67\eta + 45\eta^2 - 12\eta^3 + \eta^4) + \eta(17280 + 6720\eta - 3367\eta^2 + 4\eta^5 - \eta^6))$$

Where

$$K_1 = -0.9873623220546068;$$

and

$$K_2 = -0.000095395602883049 48;$$

We obtain the approximate solution of the second order as :

$$w(\eta) = -1636.0020206845597 + 0.000032238239253077553\eta^9 - 116.88592807205187\eta^2 + 2.9246530650391955\eta^4 + 0.48785969966796544\eta^5 + 0.032374372630945884\eta^6 + 0.9748843550130653e^{\eta}(-6.460588475454943 + \eta)(-5.454487464776046 + \eta)(47.62167957157959 - 10.05922734040209\eta + \eta^2)$$

Comparison between the absolute errors of the first order and second order are shown in the following table

Table 1. Comparison of absolute errors at different orders of approximations".

η	Exact Solution	First Order	Second Order
0.0	0.0	0.0	2.27374×10^{-13}
0.1	9.94654×10^{-2}	9.54924×10^{-5}	7.36235×10^{-8}
0.2	1.95424×10^{-1}	4.11883×10^{-4}	1.44256×10^{-7}
0.3	2.8347×10^{-1}	9.97088×10^{-4}	1.79921×10^{-8}
0.4	3.58038×10^{-1}	1.89223×10^{-3}	5.87428×10^{-7}
0.5	4.1218×10^{-1}	3.11232×10^{-2}	1.60543×10^{-6}
0.6	4.37309×10^{-1}	4.62242×10^{-2}	2.93892×10^{-6}
0.7	4.22888×10^{-1}	6.31257×10^{-2}	4.30696×10^{-6}
0.8	3.56087×10^{-1}	7.97586×10^{-2}	5.38493×10^{-6}
0.9	2.21364×10^{-1}	9.29507×10^{-2}	5.96232×10^{-6}
1.0	0.0	9.84465×10^{-2}	6.09036×10^{-6}

4. Conclusion

The study demonstrates that the Optimal Homotopy Asymptotic Method (OHAM) is a highly effective technique for solving a broad range of differential equations, encompassing both linear and nonlinear types. The method's rapid convergence to accurate solutions underscores its efficacy and practicality in computational mathematics. These findings suggest that OHAM could serve as a robust tool for addressing complex differential equations in various scientific and engineering disciplines. Future research should explore the method's adaptability to more intricate problem classes and its potential integration with other computational techniques to enhance solution accuracy and efficiency further. Additionally, empirical studies on OHAM's performance across diverse applications could provide deeper insights into its capabilities and limitations.

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