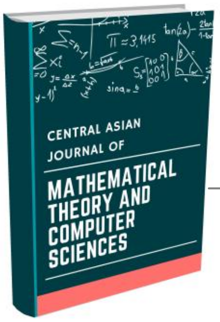

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A Proof of the Gauss - Bonnet Theorem

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ABSTRACT

The paper proposes a new proof of the well-known Gauss-Bonnet theorem, based only on metric considerations. The theorem is first proved for triangles lying in the injectivity domain of the exponential mapping, and then, using standard techniques, it is extended to arbitrary domains.

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The paper proposes a new proof of the well-known Gauss-Bonnet theorem, based only on metric considerations. The theorem is first proved for triangles lying in the exponential mapping injectivity domain, and then, using standard techniques, is extended to arbitrary domains. This approach advantage over the standard Gauss - Bonnet theorem proofs lies in its extending possibility to the multidimensional case in an inequality form in which the integral boundary curvature and some integrals over the cones stretched on the curvature boundary tensor of the multidimensional Riemannian space will participate.

Preliminary lemmas.

Let ***F*** *twice continuously differentiable surface in* $\mathbb{R}^3$, $K(M)$ *- its Gaussian curvature, where point is* $M \in F$. *The shortest connecting points A and B of the surface will be denoted by AB, the same symbol will also denote the distance on the surface* ***F*** *between points A and B.*

Lemma 1. Let $`(\rho,\varphi)'$ *- polar coordinate system on surface* ***F*** *centered at point* $O \in F$. $K(\rho,\varphi)$ *Gaussian curvature of surface* ***F****, given as a function of* ρ,φ. *Then*

$$ds = dr^2 + g(\rho,\varphi)d\varphi^2 \qquad (1.1)$$

And the function $g(\rho,\varphi)$ *is a solution to the Jacobi equation:*

$$g''_{\rho^2} + K(\rho,\varphi)g = 0 \qquad (1.2)$$

with initial conditions$g(0,\varphi) = 0,\ g'_\rho(0,\varphi) = 1$.

The proof of Lemma 1 can be found in [1] p. 175.

Lemma 2. Let (u,v)*– is a semi-geodetic coordinate system on surface* ***F****, built on geodesic* $\gamma\ (\gamma : u = 0, v = t)$. $K(u,v)$ *– Gaussian curvature of surface* ***F*** *given as a function of* $u\ ,v$. *Then*

$$ds = du^2 + f(u,v)dv^2, \qquad (1.3)$$

a function$f(u,v)$ *is a solution to the Jacobi equation:*

$$f''_{u^2} + K(u,v)f = 0 \qquad (1.4)$$

with initial conditions $f(0,v) = 1,\ f'_u(0,v) = 0$ *(see. [1] p.172).*

Lemma 3. Let $\gamma(\varepsilon)$ *- family of geodesics on surface* ***F****, whose equations in a semi-geodesic coordinate system are given by the function* $u = u(v, \varepsilon)$,

$$0 \le v \le v_0, u(v,0) = 0 \text{ and } u(v_0, \varepsilon) = \varphi(\varepsilon).$$

Then the function $\frac{\partial u}{\partial \varepsilon}|_{\varepsilon=0} = \eta(v)$ *the solution of the Jacobi equation:*

$$\eta'' + k(v)\eta = 0 \qquad (1.5)$$

with boundary conditions $\eta(0) = 0, \eta(v_0) = \varphi'(0)$.

Lemma 3 assertion. Follows from Lemma 14.3 in [2].

Proof of the Gauss-Bonnet theorem for geodesic triangles.

Theorem. *For a geodesic triangle, with interior angles* α, β, γ *and Gaussian curvature K, it is true:*

$$\alpha + \beta + \gamma = \pi + \iint K ds.$$

Let (ABC, abc) full surface triangle ***F***, made up of shortest paths. *Let* P(z) *denote a point on CB side at a distance* z *from point C. Suppose that for any* z $(0 \le z \le z_0)$ *there is only one shortest path* $AP(z)$, *smoothly depending on* z. *Let* $\alpha(z), \gamma(z)$ *and* β *vertex angles* A, $P(z)$ *and* B, *triangles* $AP(z)B$. *Then* $\gamma(a) = \pi, \alpha(a) = 0$ *and* $\alpha(z_0) - \alpha(z) = \tau(z)$. *Find the derivatives* $\frac{d\alpha(z)}{dz}$ *u* $\frac{d\gamma(z)}{dz}$ *at an arbitrary point* z_0. *Let us introduce along the shortest* $AP(z)$ *semi-geodesic coordinate system [3]. Let the equation of the shortest path* $P(z)B$ *in this coordinate system be given by the functions:* $u = u(z), v = v(z)$.

Let us denote by $u = h(v, z)$ *equation shortest* $AP(z)$, *where* $z \in [z_0, z_0 + \varepsilon)$ *then* $h(v(z), z) = u(z)$ *and*

$$\frac{dh}{dv}\frac{dv}{dz}|_{z=z_0} + \frac{\partial h}{\partial z}|_{z=z_0} = \frac{du}{dz}|_{z=z_0} \qquad (2.1)$$

Thus, $\frac{\partial h}{\partial z}|_{z=z_0} = \eta(v)$ *is the solution of the Jacobi equation (1.5) with the boundary conditions* $(v) = 0,\ \eta(v(z_0)) = \sin\gamma(z_0)$. *So*

$$\eta(v) = \frac{g(v)}{g(v_0)}\sin\gamma(z_0) \qquad (2.2)$$

where $g(v)$ *- is the standard solution of the Jacobi equation (see Lemma 1.) . Because*

$$-\cos\gamma(z) = \frac{\frac{dh(v(z),z)}{dv}\frac{du}{dz} + f^2\left(u(z), v(z)\right)\frac{dv}{dz}}{\sqrt{(\frac{du}{dz})^2 + f^2(\frac{dv}{dz})^2}\sqrt{(\frac{du}{dz})^2 + f^2}},$$

$$\cos\tau(z) = \frac{<(0,1),(h'_v,1)>}{\sqrt{f^2}\sqrt{(h'_v)^2 + f^2}}$$

hence

$$\sin\gamma \cdot \gamma'_{z_0} = \frac{d^2h}{dv^2}\frac{dv}{dz}\frac{du}{dz} + \eta'_v\frac{du}{dz} + \frac{dh}{dv}\frac{d^2u}{dz^2} + 2f\left(\frac{\partial f}{\partial u}\frac{du}{dz} + \frac{\partial f}{\partial v}\frac{dv}{dz}\right)\frac{dv}{dz} + f^2\frac{d^2v}{dz^2}$$

and

$$\sin\tau(z) = \frac{h'_v}{\sqrt{(h'_v) + f^2}},$$

or

$$\gamma'_v = \eta'_v(v(z_0)) \quad и \quad \tau'_{z_0} = \eta'_v(0) \qquad (2.3)$$

Taking into account formula (2.2) and Lemma 1., we obtain that

$$\gamma'_{z_0} = \frac{\sin\gamma(z_0)}{g(v(z_0))}\left[1 - \int_0^{v(z_0)} k(v,z)g(v,z)dv\right] \qquad (2.4)$$

and

$$\tau'_{z_0} = \frac{\sin\gamma(z_0)}{g(v(z_0))} \qquad (2.5)$$

By definition of the angle $\tau(z)$ *we have:* $\int_0^a \tau'_{z_0} dz_0 = -\int_0^a \frac{d\alpha}{dz} dz_0 = \alpha,$

therefore, from formula (2.5) we obtain:

$$\int_0^a \frac{\sin\gamma(z_0)}{g(v(z_0))} dz_0 = \alpha \qquad u$$

$$\int_0^a \frac{d\alpha}{dz} dz_0 = -\int_0^a \frac{\sin\gamma(z_0)}{g(v(z_0))} dz_0 \qquad (2.6)$$

Since the angle $\alpha(z_0)$ *and the distance* $v(z_0)$ *satisfy the condition of Lemma 1., integrating* γ'_{z_0} *over z from zero to a, and taking into account (2.6) from (2.4) we obtain*

$$\gamma(a) - \gamma(0) = \alpha - \int_0^a \int_0^{v(z_0)} k(v,z)g(v,z)dv\frac{\sin\gamma(z_0)}{g(v(z_0))}dz_0 = \alpha +$$

$$+\int_{\alpha}^{0}\int_{0}^{v(z_0)} k(v,z)g(v,z)dvd\alpha = \alpha - \int_{0}^{\alpha}\int_{0}^{v(z_0)} k(v,z)ds$$

or equivalently $\alpha + \beta + \gamma = \pi + \iint Kds$.

Thus, we have obtained a formula, which is called the Gauss-Bonnet formula for geodesic triangles.

General case of the Gauss - Bonnet theorem.

We restrict ourselves to the case of a domain homeomorphic to a disc of a bounded piecewise regular closed curve. Let a γ -piecewise regular (piecewise twice continuously differentiable) curve bounding a domain D of a complete surface **F** homeomorphic to a disc. We denote by γ_i the links (i=1,…,k), and by β_i the angles between the links γ_{i-1} and $\gamma_i$from the side of the domain D. Then the following formula is valid (Gauss - Bonnet theorem):

$$\sum_{i=1}^{k}\int_{\gamma_i} k_g ds + \sum_{i=2}^{k+1}(\pi - \beta_i)ds + \iint_D Kds = 2\pi,$$

(3.1)

Here k_g- is a geodesic curvature γ , and K is Gaussian curvature of surface **F**.

Let us give a brief description of this formula deriving idea.

Let γ_n be polygons sequence composed of shortest paths of the surface **F** converging to γ. We denote by $A_1, A_2, \dots, A_n, A_{n+1} = A_1$ the polygon vertices γ_n, and by α_i^n the angle between the links $A_{i-1}A_i$ and A_iA_{i+1} from the side of the domain D_n bounded by the polygon γ_n. Then

$$\lim_{n\to\infty}\sum_{i=1}^{n+1}(\pi - \alpha_i^n) = \sum_{i=1}^{k}\int_{\gamma_i} k_g\, ds + \sum_{i=2}^{k+1}(\pi - \beta_i)$$

(3.2)

The existence of polygons sequence with the indicated properties is proved, for example, in [4] p. 214. We divide the domain D_n *into triangles so small that one can apply the theorem from §2 to each of them. For this, it is sufficient to require that each triangle diameter be less than the injectivity radius of* ***F****. Let* $N(n)$ *denote the number of these triangles:* $\Delta_1^n, \Delta_2^n, \dots, \Delta_{N(n)}^n$, *and after* $\alpha_j^n, \beta_j^n, \gamma_j^n$ *- interior corners of triangle* Δ_j^n *. Then, by theorem §2, we obtain*

$$\alpha_j^n + \beta_j^n + \gamma_j^n = \pi + \iint_{\Delta_j^n} kds \qquad (3.3)$$

Let's count the number of all triangles sides included in the partition of D_n *area. The number of vertices lying on the border of the area* D_n*is equal to* n*. Therefore, the number of sides is*$3N(n)/2 + n/2 = (3N(n) + n)/2$. *The number of vertices* P_N *of the partition of the region* D_n *is found from the Euler formula:* $P_N - \frac{3N+n}{2} + N = 1$ *, where* $P_N = 1 + \frac{N+n}{2}$ *and the number of internal vertices is* $P_N - n = 1 + \frac{N-n}{2}$ *. Consequently*

$$\sum_{j=1}^{N(n)}\left(\alpha_j^n + \beta_j^n + \gamma_j^n\right) = \Sigma' + 2\pi(1 + \frac{N-n}{2}) = \Sigma' + 2\pi + \pi(\Sigma' + 2\pi(N-n)$$

(3.4)

Here Σ' *is the sum of the angles at the vertices lying on the boundary of the domain* D_n*. From (3.3) and (3.4) we obtain:*

$$\Sigma' = \sum_{i=1}^{n+1}(\alpha_i^n) = n\pi - 2\pi + \iint_{D_n} kds.$$

(3.5)

Hence

$$\sum_{i=2}^{n+1}(\pi - \alpha_i^n) = 2\pi - \iint_{D_n} kds \qquad (3.6)$$

Passing in (3.6) to the limit as $n \to \infty$ and taking into account (3.2), we obtain the Gauss-Bonnet formula

$$\sum_{i=1}^{k}\left(\int_{\gamma_j} k_g ds\right) + \sum_{i=2}^{k+1}(\pi - \beta_i) + \iint_D kds = 2\pi$$

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