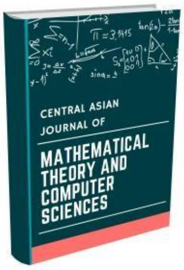

CENTRAL ASIAN JOURNAL OF MATHEMATICAL THEORY AND COMPUTER SCIENCES
https://cajmtcs.centralasianstudies.org/index.php/CAJMTCS
Volume: 05 Issue: 02 | Feb 2024 ISSN: 2660-5309

Article

On Weak Parallelogram Constants for Lebesgue Spaces

MhamdHasn Abdulrzaq[1], **MhamdHasn Allatef**[1]

1. Basra, Iraq
* Correspondence: madmoh3791@gmail.com

Abstract: This paper comprehensively examines fundamental principles pertaining to parallelogram and parallelogram (abbreviated as w.p.) laws. Emphasis is placed on investigating the duality in the w.p. laws, alongside a thorough review and analysis of foundational findings concerning the geometry of w.p. spaces. Additionally, the research delves into the exploration of optimal w.p. constants for Lebesgue spaces. Through rigorous examination and synthesis of pertinent literature, this study aims to provide a comprehensive understanding of the theoretical underpinnings and practical implications of parallelogram laws and their application in Lebesgue spaces.

Keywords: Normed space, Lebesgue space, Parallelogram Laws, Smooth Norm, Uniformly Smooth Norm, Weak Parallelogram Constant

1. Introduction

Parallelogram laws play a fundamental role in functional analysis, particularly in the study of Lebesgue spaces. These laws provide essential insights into the geometric structure and properties of normed vector spaces. Originating from Euclidean geometry, these laws are applied to linear spaces. In the context of Lebesgue spaces, understanding the laws becomes particularly important due to their role in the study of integration and functions. This introduction aims to explore the fundamentals of parallelogram laws in functional analysis, with a focus on their role in Lebesgue spaces and practical applications.

Lebesgue spaces were first introduced in 1902 by Henry Lebesgue [1], and soon became the center of much research. In 1967, D. Kay [2] explored parallelogram law in a number of L p space. In 1976, W. Bynum [3] obtained an equivalent number characterizations for Banach spaces satisfying a couple of weak parallelogram laws. In 1991, Zong-Ben Xu, G. F. Roach [4], gave some inequalities in uniformly smooth and convex spaces. In 2003, R. Cheng, A. G. Miamee and M. Pourahmadi [5] derived a Pythagorean theorem for L_p spaces. In 2013, R. Cheng and C. B. Harris [6], obtained a number of properties regarding a family of weak parallelogram laws in Banach space. Also in 2015, R. Cheng and W. Ross [7] further discussed them in a Banach space, showing their applicability to Lebesgue spaces and connections to basic geometric principles.

Citation: Abdulrzaq, M., & Allatef, M. On Weak Parallelogram Constants for Lebesgue Spaces. Central Asian Journal of Mathematical Theory and Computer Sciences 2024, 5(2), 36-51.

Received: 2nd March 2024
Revised: 11th March 2024
Accepted: 18th March 2024
Published: 25th March 2024

2. Materials and Methods

2.1. Definition of Weak Parallelogram Constants

The study begins by defining weak parallelogram constants and their significance in the context of Lebesgue spaces. These constants quantify the deviation from parallelogram law in normed spaces, providing insights into the geometry and structure of these spaces.

2.2. Analysis of Lebesgue Spaces

The investigation focuses on Lebesgue spaces, which are fundamental in functional analysis and measure theory. The study examines the properties and structure of Lebesgue spaces, highlighting their relevance to the study of weak parallelogram constants.

2.3. Computation and Derivation

Mathematical techniques are employed to compute and derive weak parallelogram constants for various Lebesgue spaces. This involves analyzing integral inequalities, functional forms, and other mathematical structures inherent to Lebesgue spaces.

2.4. Comparative Study

The study compares weak parallelogram constants across different Lebesgue spaces, identifying patterns, trends, and variations. This comparative analysis provides insights into the behavior of these constants under different conditions and parameters.

3. Results

3.1. Basics

Definition 1.1. [8] Let $\mathcal{Z}$ be a vector space on the set of complex numbers. A mapping $\|\cdot\|: \mathcal{Z} \to [0, \infty)$ is called a norm on $\mathcal{Z}$ if for every $a, b \in \mathcal{Z}$ and $\alpha \in \mathbb{C}$,

- $\| a \| = 0$ if and only if $a = 0$,
- $\| \alpha a \| = |\alpha| \, \| a \|$,
- $\| a + b \| \leq \| a \| + \| b \|$.

In this case, the pair $(\mathcal{Z}, \|\cdot\|)$ or simply $\mathcal{Z}$ is called a normed space.

Definition 1.2. [8] A sequence (ξ_n) in a normed space $\mathcal{Z}$ is called Cauchy if $\lim_{m,n\to\infty} \|\xi_m - \xi_n\| = 0$.

Definition 1.3. [8] A normed space $\mathcal{Z}$ is a Banach space if for any Cauchy sequence $(\xi_n) \subseteq \mathcal{Z}, \exists \xi \in \mathcal{Z}$ s.t. $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$. Example 1.1.4. A topological space X is compact whenever collection $\{U_\alpha\}$ of open subsets of X, if

$$X = \bigcup_{\alpha} U_\alpha$$

there are $\alpha_1, \dots, \alpha_n$ such that

$$X = \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Assume X is compact, and the space of complex-valued continuous functions defined on X is denoted by $C(X)$. Then, $C(X)$ equipped with the norm

$$\| h \|_{\sup} := \sup_{\xi \in X} |h(\xi)|$$

for all $h \in C(X)$, and operations

$$(f+g)\,(\xi) := f(\xi) + g(\xi),\ (\xi \in X)$$

and

$$(\alpha f)(\xi) := \alpha f(\xi),\ (\xi \in X)$$

for all $f, g \in C(X)$ and $\alpha \in \mathbb{C}$, is a Banach space.

Example 1.4. Let $n \geq 2$ be a fixed natural number, and $p \geq 1$. Then, $\mathbb{C}^n$ equipped with

$$\|(\xi_1, \ldots, \xi_n)\| := (|\xi_1|^p + \cdots + |\xi_n|^p)^{\frac{1}{p}}$$

and operations

$$(\xi_1 + \cdots + \xi_n) + (\eta_1, \ldots, \eta_n) := (\xi_1 + \eta_1, \ldots, \xi_n + \eta_n)$$

and

$$\alpha(\xi_1, \ldots, \xi_n) := (\alpha\xi_1, \ldots, \alpha\xi_n)$$

for all $(\xi_1, \ldots, \xi_n), (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, is a Banach space. Definition 1.1.6. Let $\mathcal{Z}$ and $\mathcal{Y}$ be Banach spaces. Then a mapping $T: \mathcal{Z} \to \mathcal{Y}$ is called a bounded operator if

$$\| T \| := \sup\{\| T(\xi) \| : \| \xi \| \leq 1\} < \infty$$

The bounded family of linear operators from $\mathcal{Z}$ into $\mathcal{Y}$ is denoted by $B(\mathcal{Z}, \mathcal{Y})$.

The set $\mathbb{C}$ with the usual operations and the absolute value mapping $|\cdot|$ as a norm, is a Banach space. Put $\mathcal{Z}^* := B(\mathcal{Z}, \mathbb{C})$, and call it the $dual$ of $\mathcal{Z}$. So, for each $\xi^* \in \mathcal{Z}^*$,

$$\|\xi^*\| = \sup\{|\xi^*(\xi)| : \| \xi \| \leq 1\}$$

Definition 1.5. [9] Let V be a linear space with the scalars belong $\mathbb{C}$. A mapping

$$\langle \cdot,\cdot \rangle : V \times V \to \mathbb{C}$$

is called an inner product on V if:

1) for every $\xi \in V, \langle \xi, \xi \rangle \geq 0$;

2) for every $\xi, \eta, z \in V$ and $\alpha \in \mathbb{C}$ we have

$$\langle \alpha\xi + \eta, z \rangle = \alpha\langle \xi, z \rangle + \langle \eta, z \rangle$$

3) to every $\xi, \eta \in V, \overline{\langle \xi, \eta \rangle} = \langle \eta, \xi \rangle$.

If the above applies, V is called an inner product space.

Remark 1.6. Let V be a linear space equipped with the inner $\langle \cdot,\cdot \rangle$. Then, the mapping $\|\cdot\|$ on V defined by

$$\| \xi \| := \langle \xi, \xi \rangle^{\frac{1}{2}}$$

for all $\xi \in V,$ is a norm on V which induced by the inner product.

Definition 1.7. [9] A pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is said to be a Hilbert space if $\mathcal{H}$ equipped with the norm defined in Remark 1.6 induced by its inner product to be Banach.

Example 1.8. [9] $\mathbb{C}^2 := \mathbb{C} \times \mathbb{C}$ equipped the inner product

$$\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle := \xi_1 \overline{\xi_2} + \eta_1 \overline{\eta_2}$$

is a Hilbert space.

Definition 1.9. [10] $\mathcal{A}$ as a family of subsets of the non-empty set X called σ-algebra if $\emptyset \in \mathcal{A}$ and

- for each $E \in \mathcal{A}, E^c := X \setminus E \in \mathcal{A},$
- for each $E_1, E_2, \ldots$ in $\mathcal{A}$ we have $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}.$

In this case, the pair $(X, \mathcal{A})$ is said to be a measurable space.

Definition 1.10. [10] Assume $(X, \mathcal{A})$ is a measurable space. Then, $f : X \to \mathbb{C}$ is called measurable if for each open subset $E \subseteq \mathbb{C}$ we have $f^{-1}(E) \in \mathcal{A}.$

Definition 1.11. [11] Let $X \neq \emptyset$ be a set and $\mathcal{A}$ be a σ-algebra on X. Assume that $\mu: \mathcal{A} \to [0, \infty]$ satisfies:

1) $\mu(\emptyset) = 0,$
2) for each disjoint elements $E_1, E_2, \ldots \in \mathcal{A},$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

In this case, μ is called a measure on X, and $(X, \mathcal{A}, \mu)$ a measure space.

Definition 1.12. [11] suppose (X, Ω, μ) is a measure space. Assume $0 < p.$ to each measurable function $h: X \to \mathbb{C}$ put

$$\| h \|_p := \left(\int_X |h|^p d\mu \right)^{\frac{1}{p}}.$$

Then, we define the Lebesgue space by

$$L_p = L_p(X, \Omega, \mu) := \{ h: X \to \mathbb{C}: h \text{ is measurable and } \| h \|_p < \infty \}.$$

Elements $f, h \in L_p$ would be considered the same if $f = h$ almost everywhere i.e. $\mu(\{\xi \in X: f(\xi) \neq h(\xi)\}) = 0.$

Definition 1.13. [11] suppose (X, Ω, μ) is a measure space. For each measurable function $h: X \to \mathbb{C}$ we put

$$\| h \|_\infty := \inf\{c > 0: |h| \leq c \text{ a.e. }\}.$$

Then, the collection of complex-valued measurable functions h on X with $\| h \|_\infty < \infty$, is denoted by $L_\infty(X, \mathcal{A}, \mu)$ or simply L_∞. In fact, $f \in L_\infty$ if and only there is a number $c > 0$ s.t. $\mu(A) = 0$, where

$$A := \{\xi \in X : |h(\xi)| > c\}.$$

Here, also we assume that two functions $f, h \in L_\infty$ are the same if $f = h$ a.e.

Theorem 1.14. [12] If $1 \leq p \leq \infty$, then the normed space $\left(L_p, \|\cdot\|_p\right)$ is Banach.

Proof. See [12].

Remark 1.15. [9] $L_2(X, \mathcal{A}, \mu)$ is a Hilbert space. In fact, $\|\cdot\|_2$ is induced by

$$\langle f, g \rangle := \int_X f \bar{g} d\mu.$$

Remark 1.16. [8] Let $p > 0$. Let m be a measure defined on the power set of $\mathbb{N}$ by $m(E) :=$ the number of elements of E if $E \subseteq \mathbb{N}$ is finite, and $m(E) := \infty$ if E is not finite. This measure is called the counting measure. In this case, trivially every function $f : \mathbb{N} \to \mathbb{C}$ is measurable, and f can be considered as a sequence $(a_n)_{n=1}^{\infty}$, where $f(n) := a_n$. Then, we have

$$\int_{\mathbb{N}} |f|^p dm = \sum_{n=1}^{\infty} |a_n|^p$$

This means that

$$\ell_p := L_p(\mathbb{N}, P(\mathbb{N}), m) = \left\{(a_n) \subseteq \mathbb{C} : \sum_{n=1}^{\infty} |a_n|^p < \infty\right\}.$$

The following are some relevant parallelogram laws

Definition 1.17. [8] suppose $\mathcal{Z}$ is a Banach space. Let $C > 0$ and $r \in (1, \infty)$.

- $\mathcal{Z}$ satisfies r-lower w.p. law with the constant C (simply $\mathcal{Z}$ is r-LWP (C)) if for each $\xi, \eta \in \mathcal{Z}$,

$$\| \xi + \eta \|^r + C \| \xi - \eta \|^r \leq 2^{r-1}(\| \xi \|^r + \| \eta \|^r)$$

- $\mathcal{Z}$ satisfies r-upper w.p. law with the constant C (simply $\mathcal{Z}$ is $r - \mathrm{UWP}(C)$) if for each $\xi, \eta \in \mathcal{Z}$,

$$\| \xi + \eta \|^r + C \| \xi - \eta \|^r \geq 2^{r-1}(\| \xi \|^r + \| \eta \|^r)$$

The constant C in the above relations is called the w.p. constant.

Remark 1.18. If $\mathcal{Z}$ is a r-LWP (C) space, then by putting $\eta = -\xi$ in relation (1.17) we have

$$C(2^r \| \xi \|^r) \leq 2^{r-1}(\| \xi \|^r + \| \xi \|^r)$$

So $0 < C \leq 1$. Similarly, if $\mathcal{Z}$ is a $r-\mathrm{UWP}\ (\mathrm{C})$ space we have $C \geq 1$, because if in (1.18) we replace η by $-\xi$, then we have

$$C2^{r} \parallel \xi \parallel^{r} \geq 2^{r} \parallel \xi \parallel^{r}$$

Recall that an inner product space $\mathcal{H}$ is a Hilbert space if it is complete

Theorem 1.19. Any complex Hilbert space $\mathcal{H}$ satisfies the following which is caled Parallelogram law:

$$\parallel \xi + \eta \parallel^{2} + \parallel \xi - \eta \parallel^{2} = 2(\parallel \xi \parallel^{2} + \parallel \eta \parallel^{2})$$

for all $\xi, \eta \in \mathcal{H}$.

By [13] and [14], Ch. 7, if $\mathcal{Z}$ is a Banach space and satisfies the Parallelogram Law, then it is a Hilbert space, and

$$\langle \xi, \eta \rangle := \frac{1}{4}(\parallel \xi + \eta \parallel^{2} + i \parallel \xi + i\eta \parallel^{2} - \parallel \xi - \eta \parallel^{2} - i \parallel \xi - i\eta \parallel^{2}).$$

In fact, it is possible to show that if (1.19) is weaken to

$$\parallel \xi + \eta \parallel^{2} + \parallel \xi - \eta \parallel^{2} \leq 2(\parallel \xi \parallel^{2} + \parallel \eta \parallel^{2}),\ (\xi, \eta \in \mathcal{Z})$$

then $\mathcal{Z}$ is also a Hilbert space.

3.2. Lebesgue spaces and w.p. spaces

The following are some results that we will prove regarding a Lebesgue space

Theorem 2.1. Suppose $\mathrm{p} \in (1,2]$, and further that p' is its the conjugate exponent, then $\mathrm{L_p}$ is:

$$\begin{aligned} &\mathrm{r-UWP}\ (1) && \text{if } \mathrm{r} \in (1, \mathrm{p}]; \\ &\mathrm{r-LWP}\ \left((\mathrm{p}-1)^{\mathrm{r}/2}\right) && \text{if } \mathrm{r} \in [2, \mathrm{p}'];\ \text{and} \\ &\mathrm{r-LWP}\ (1) && \text{if } \mathrm{p}' \leq \mathrm{r} \end{aligned}$$

when $\mathrm{p} \in [2, \infty)$, p' is the conjugate exponent, then $\mathrm{L_p}$ is:

$$\begin{aligned} &\mathrm{r-LWP}\ (1) && \text{if } \mathrm{r} \in (\mathrm{p}, \infty) \\ &\mathrm{r-UWP}\ \left((\mathrm{p}-1)^{\mathrm{r}/2}\right) && \text{if } \mathrm{r} \in [\mathrm{p}', 2];\ \text{and} \\ &\mathrm{r-UWP}\ (1) && \text{if } \mathrm{p}' \geq \mathrm{r} \end{aligned}$$

Lemma 2.2. Assume $\mathrm{p}, \mathrm{r} \in (1, \infty)$.

(i) If $\mathrm{r} \geq \max\{\mathrm{p}, \mathrm{p}'\}$, we have for each $\eta, \zeta \in \mathrm{L_p}$,

$$\parallel \eta - \zeta \parallel_{\mathrm{p}}^{\mathrm{r}} + \parallel \eta + \zeta \parallel_{\mathrm{p}}^{\mathrm{r}} \leq 2^{\mathrm{r}-1}\left(\parallel \eta \parallel_{\mathrm{p}}^{\mathrm{r}} + \parallel \zeta \parallel_{\mathrm{p}}^{\mathrm{r}}\right)$$

(ii) If $\mathrm{r} \leq \min\{\mathrm{p}, \mathrm{p}'\}$, then for every $\eta, \zeta \in \mathrm{L_p}$,

$$\parallel \eta - \zeta \parallel_{\mathrm{p}}^{\mathrm{r}} + \parallel \eta + \zeta \parallel_{\mathrm{p}}^{\mathrm{r}} \geq 2^{\mathrm{r}-1}\left(\parallel \eta \parallel_{\mathrm{p}}^{\mathrm{r}} + \parallel \zeta \parallel_{\mathrm{p}}^{\mathrm{r}}\right)$$

(iii) If $1 < \mathrm{p} \leq \mathrm{r} \leq \mathrm{p}' < \infty$, then for every $\eta, \zeta \in \mathrm{L_p}$,

$$\parallel \eta + \zeta \parallel_{\mathrm{p}}^{\mathrm{r}} + \parallel \eta - \zeta \parallel_{\mathrm{p}}^{\mathrm{r}} \leq 2^{\mathrm{r}/\mathrm{p}}\left(\parallel \eta \parallel_{\mathrm{p}}^{\mathrm{r}} + \parallel \zeta \parallel_{\mathrm{p}}^{\mathrm{r}}\right)$$

(iv) If $1 < p' \leq r \leq p < \infty$, then for every $\eta, \zeta \in L_p$,

$$\| \eta - \zeta \|_p^r + \| \eta + \zeta \|_p^r \geq 2^{r/p} \left(\| \eta \|_p^r + \| \zeta \|_p^r \right)$$

Proof. Note that for every $2 \leq r \leq p < \infty$,

$$\begin{aligned} & \left(\| \eta + \zeta \|_p^r + \| \eta - \zeta \|_p^r \right)^{p/r} \\ & \leq \| \eta + \zeta \|_p^p + \| \eta - \zeta \|_p^p \\ & \leq 2^{p-1} \left(\| \eta \|_p^p + \| \zeta \|_p^p \right) \\ & = 2^{p-1} \left(\| \eta \|_p^p \cdot 1 + \| \zeta \|_p^p \cdot 1 \right) \\ & \leq 2^{p-1} \left(\| \eta \|_p^r + \| \zeta \|_p^r \right)^{p/r} \left(1^{\frac{r}{r-p}} + 1^{\frac{r}{r-p}} \right)^{\frac{r-p}{r}}. \end{aligned}$$

Then, we conclude that:

$$\frac{r(p-1)}{p} + \frac{r(r-p)}{rp} = (r-1)$$

So, (i) holds. For every $2 \leq p' \leq r < \infty$, we have

$$\begin{aligned} 2 \quad & \left(\| \eta \|_p^r + \| \zeta \|_p^r \right)^{\frac{p'(p-1)}{r}} \\ & \leq 2 \left(\| \eta \|_p^{p'} + \| \zeta \|_p^{p'} \right)^{p-1} \\ & \leq \| \eta + \zeta \|_p^p + \| \eta - \zeta \|_p^p \\ & \leq \left(\| \eta + \zeta \|_p^r + \| \eta - \zeta \|_p^r \right)^{\frac{p}{r}} 2^{1-\frac{p}{r}}. \end{aligned}$$

Hence, $2\left(\| \eta \|_p^r + \| \zeta \|_p^r \right)$ is less than $\| \eta + \zeta \|_p^r + \| \eta - \zeta \|_p^r$.

If we change η and ζ to $\eta + \zeta$ and $\eta - \zeta$, respectively, the proof of (i) will be complete. The other parts are proved similarly.

Lemma 2.3. Assume $r, p \in (1, \infty)$.

(i) If $p \in [r, 2]$, then for all $\eta, \xi \in L_p$,

$$\| \eta + \xi \|_p^r + (p-1)^{r/2} \| \eta - \xi \|_p^r \leq 2\left(\| \eta \|_p^r + \| \xi \|_p^r \right)$$

(ii) If $1 < p \leq 2 \leq r \leq p'$, then for every η and ξ in L_p,

$$\| \eta + \xi \|_p^r + (p-1)^{r/2} \| \eta - \xi \|_p^r \leq 2^{r-1}\left(\| \eta \|_p^r + \| \xi \|_p^r \right)$$

(iii) Given $r \in [2, p]$ and $p < \infty$, then for any $\eta, \xi \in L_p$,

$$\| \eta + \xi \|_p^r + (p-1)^{r/2} \| \eta - \xi \|_p^r \geq 2\left(\| \eta \|_p^r + \| \xi \|_p^r \right)$$

(iv) If $p \in [2, \infty)$ and $r \in [p', 2]$, then for every η and ξ in L_p,

$$\| \eta + \xi \|_p^r + (p-1)^{r/2} \| \eta - \xi \|_p^r \geq 2^{r-1}\left(\| \eta \|_p^r + \| \xi \|_p^r \right)$$

Proof. For the proof of (i), apply (3.1.5) for the space L_p to get

$$\begin{aligned} \| \eta \quad + \xi \Big\|_p^r + (p-1)^{\frac{r}{2}} \Big\| \eta - \xi \|_p^r \\ &\le \left(\| \eta + \xi \|_p^2 + (p-1) \| \eta - \xi \|_p^2 \right)^{\frac{r}{2}} (1+1)^{1-\frac{r}{2}} \\ &\le \left(2\left[\| \eta \|_p^2 + \| \xi \|_p^2 \right] \right)^{\frac{r}{2}} 2^{1-\frac{r}{2}} \\ &\le 2\left(\| \eta \|_p^r + \| \xi \|_p^r \right) \end{aligned}$$

The proof of others are similar.

The following are some results regarding w.p. spaces.

Definition 2.4. [15] $\mathcal{Z}$ is called uniformly smooth whenever for any $\varepsilon > 0$ there is $\delta > 0$ s.t. for each $x, \xi \in \mathcal{Z}$ with $\| x \| = 1$ and $\| \xi \| \le \delta$ we have

$$\| x + \xi \| + \| x - \xi \| \le 2 + \varepsilon \| \xi \|.$$

Definition 2.5. [15] A normed space $\mathcal{Z}$ is uniformly convex if for each $\varepsilon \in (0,2]$ $\exists$ $\delta > 0$ s.t. for each $x, \xi \in \mathcal{Z}$, if $\| x \| = \| \xi \| = 1$ and $\| x - \xi \| \ge \varepsilon$, then

$$\left\| \frac{x + \xi}{2} \right\| \le 1 - \delta.$$

Proposition 2.6. [15] Let $p \in (1, \infty)$. If $\mathcal{Z}$ has the property $p - \mathrm{LWP(C)}$, then $\mathcal{Z}$ is uniformly convex. If $\mathcal{Z}$ has the property $p - \mathrm{UWP(C)}$, then $\mathcal{Z}$ is uniformly smooth.

Remark 2.7. The modulus of convexity in the Banach space V is a map $\delta_V(t)$ defined for $t \in [0,2]$ by

$$\delta_V(t) := \inf\{1 - (\| \zeta + \eta \|/2) : (\zeta, \eta) \in A_t\}$$

where A_t is the set of all pairs $(\zeta, \eta) \in V \times V$ s.t. $\| \zeta \| = 1, \| \eta \| = 1$, and $\| \zeta - \eta \| = t$. And a function $\rho_V(t)$ defined for $t \ge 0$ by

$$\rho_V(t) := \frac{1}{2} \sup\{\| \zeta + t\eta \| + \| \zeta - t\eta \| - 2 : \| \zeta \| = \| \eta \| = 1\}$$

is called the modulus of smoothness of V.

Recall also that V is uniformly convex if $\delta_V(t) > 0$ for every $t \in (0,2]$, and V is called uniformly smooth if the right derivative of ρ_V at 0 is 0 . We need the below fact in the next proof.

Lemma 2.8. [16] Assume that X is a normed space, and $T \in X^*$. Assume that $\zeta, \eta \in X$ and $\| \eta \| = 1$ and $T(\eta) = \| T \|$. Also, assume that there is $p \ge 1$ s.t. the limit exist.

$$\lim_{t \to 0} \frac{\| \eta + t\zeta \|^p - \| \eta \|^p}{pt}$$

Then, we have

$$T(\zeta) = \| T \| \lim_{t \to 0} \frac{\| \eta + t\zeta \|^p - \| \eta \|^p}{pt}$$

Proof. See [16] Lemma 11.17.

Lemma 2.9. suppose $q \in (1, \infty)$. Suppose further that $\mathcal{Z}$ is a smooth Banach space. $\mathcal{Z}$ is q-LWP iff for some $K > 0$,

$$\| \zeta + \eta \|^q \geq \| \zeta \|^q + K \| \eta \|^q + q \| \zeta \|^{q-1} \Re\big(T_\zeta(\eta)\big)$$

for all $\zeta \neq 0$ and $\eta \in \mathcal{Z}$. Also, $\mathcal{Z}$ is $q - \mathrm{UWP}$ iff there is a $K > 0$ such that for every $\zeta \neq 0$ and $\eta \in \mathcal{Z}$,

$$\| \zeta + \eta \|^q \leq \| \zeta \|^q + K \| \eta \|^q + q \| \zeta \|^{q-1\Re\left(T_\zeta(\eta)\right)}$$

Proof. Suppose (2.4.) hold. So:

$$\| \zeta \|^q + K \| \pm\eta \|^q + q \| \zeta \|^{q-1} \Re\big(T_\zeta(\pm\eta)\big) \leq \| \zeta \pm \eta \|^q$$

and so

$$\| \zeta \|^q + K \| \eta \|^q \pm q \| \zeta \|^{q-1} \Re\big(T_\zeta(\eta)\big) \leq \| \zeta \pm \eta \|^q.$$

If we add the above two cases, we get

$$2 \| \zeta \|^q + 2K \| \eta \|^q \leq \| \zeta + \eta \|^q + \| \zeta - \eta \|^q$$

and so

Replace ζ with $\zeta + \eta$, and replace η with $\zeta - \eta$, to conclude that $\| \zeta + \eta \|^q + K \| \zeta - \eta \|^q$ is less than or equal to $2^{q-1}(\| \zeta \|^q + \| \eta \|^q)$. Thus $\mathcal{Z}$ is $q - \mathrm{LWP(K)}$.

Conversely, let there is $C > 0$ s.t. $\mathcal{Z}$ be $q - \mathrm{LWP}\ (C)$, and note that $\| u + v \|^q + C \| u - v \|^q$ is less than or equal to $2^{q-1}(\| u \|^q + \| v \|^q)$.

Applying this inequality for $u = \zeta$ and $v = \zeta + 2^{-n}y$, for $n = 0,1,2, \ldots$ For $n = 0$,

$$\left\|\zeta + \frac{1}{2}\eta\right\|^q + C\left\|\frac{1}{2}\eta\right\|^q \leq \frac{1}{2} \| \zeta \|^q + \frac{1}{2} \| \zeta + \eta \|^q$$

Then,

$$2\|\zeta + 2^{-(n+1)}\eta\|^q - \| \zeta \|^q + 2C\|2^{-(n+1)}\eta\|^q \leq \|\zeta + 2^{-n}\eta\|^q$$

Applying (2.5.), with $n = 0,1,2, \ldots$, we have

$$\begin{aligned} \| \zeta + \eta \|^q \quad & \geq 2\|\zeta + 2^{-1}\eta\|^q - \| \zeta \|^q + 2C\|2^{-1}\eta\|^q \\ & = -(1) \| \zeta \|^q + \left[\frac{1}{2^{q-1}}\right] C \| \eta \|^q + 2^1\|\zeta + 2^{-1}\eta\|^q \\ & \geq 2\big(2\|\zeta + 2^{-2}\eta\|^q + 2C\|2^{-2}\eta\|^q - \| \zeta \|^q\big) + 2C\|2^{-1}\eta\|^q - \| \zeta \|^q \\ & = -(1 + 2) \| \zeta \|^q + \left[\frac{1}{2^{q-1}} + \frac{1}{(2^{q-1})^2}\right] C \| \eta \|^q + 2^2\|\zeta + 2^{-2}\eta\|^q. \end{aligned}$$

By repeating the equation (2.5.) we conclude that

$$\begin{aligned}\| \eta + \zeta \|^q \quad & \geq - \| \zeta \|^q \sum_{k=0}^{n-1} 2^k + C \| \eta \|^q \sum_{k=1}^{n} \frac{1}{(2^{q-1})^k} + 2^n \|\zeta + 2^{-n}\eta\|^q \\ & = -(2^n - 1) \| \zeta \|^q + \frac{1 - 2^{-(q-1)n}}{2^{q-1} - 1} C \| \eta \|^q + 2^n \|\zeta + 2^{-n}\eta\|^q \\ & = \| \zeta \|^q + \frac{1 - 2^{-(q-1)n}}{2^{q-1} - 1} C \| \eta \|^q + \frac{\|\zeta + 2^{-n}\eta\|^q - \| \zeta \|^q}{2^{-n}}\end{aligned}$$

Then, because of Lemma 2.8,

$$\lim_{t \to 0+} \frac{\| \zeta + t\eta \|^q - \| \zeta \|^q}{qt} = \frac{T_\zeta(\eta)}{\|T_\zeta\|}.$$

This implies that

$$\lim_{t \to 0+} \frac{\| \zeta + t\eta \|^q - \| \zeta \|^q}{t} = q \| \zeta \|^{q-1\Re} \left(T_\zeta(\eta)\right)$$

The conclusion is

$$\| \zeta + \eta \|^q \geq \| \zeta \|^q + \frac{\| \eta \|^q C}{2^{q-1} - 1} + q \| \zeta \|^{q-1} \Re\left(T_\zeta(\eta)\right)$$

So, (2.4) holds with $K = C/(2^{q-1} - 1)$.

Theorem 2.10. If $\mathcal{Z}$ is smooth, and it is $p - \mathrm{LWP}$ (C), then there is $K > 0$ s.t. whenever $\zeta \perp \eta$ in $\mathcal{Z}$,

$$\| \zeta \|^p + K \| \eta \|^p \leq \| \zeta + \eta \|^p$$

If $\mathcal{Z}$ is $p - \mathrm{UWP}(C)$, then there is constant $K > 0$ s.t. whenever $\zeta \perp \eta$ in $\mathcal{Z}$

$$\| \zeta \|^p + K \| \eta \|^p \geq \| \zeta + \eta \|^p$$

It could be possible that $K := C/(2^{p-1} - 1)$.

Proof. Note that $T_\zeta(\eta) = 0$ if and only if $\zeta \perp \eta$. Now, by the proof of Lemma 2.9, $K = C/(2^{p-1} - 1)$ suffices.

In the above result, K is called Pythagorean Constant.

The Lebesgue spaces are a class of some kind of spaces which the Pythagorean inequalities hold for them. Corollary 3.2.8. Let $1 < r, p < \infty$. If $p \in (1,2]$ and $r \in [2, \infty)$, or $p \in [2, r]$ and $r \in [p, \infty)$, then for some $K > 0$,

$$K \| v \|_p^r + \| u \|_p^r \leq \| u + v \|_p^r$$

whenever $u \perp v$ in L_p.

If $r \in (1, p]$ and $p \leq 2$ or $r \in [1,2]$ and $2 \leq p < \infty$, then there is $K > 0$ s.t. $\| a \|_p^r + K \| b \|_p^r$ is greater than or equal to $\| a + b \|_p^r$ whenever $a \perp b$ in L_p.

Corollary 3.2.9. Let $\mathcal{Z}$ be smooth. Assume $\{X_n\}_{n=0}^{\infty}$ is an innovation sequence of unit vectors in X, and $\sum_{n=0}^{\infty} a_n X_n$ converges with respect to the norm.

If X has the property p-LWP, then

$$\left\| \sum_{n=0}^{\infty} a_n X_n \right\|^p \geq \sum_{n=0}^{\infty} K^n |a_n|^p$$

in which K is the Pythagorean constant.

If X has the property $p - \mathrm{UWP}$, then

$$\left\| \sum_{n=0}^{\infty} a_n X_n \right\|^p \leq \sum_{n=0}^{\infty} K^n |a_n|^p$$

in which K is the Pythagorean constant.

Proposition 2.11. Let $1 < r < \infty$, and $1 < p < \infty$. If $r > p$ or $r > 2$, then there is not $C > 0$ s.t. for every η and ζ belong to L_p,

$$\| \eta + \zeta \|_p^r + C \| \eta - \zeta \|_p^r \geq 2^{r-1} \left(\| \eta \|_p^r + \| \zeta \|_p^r \right)$$

If $r < p$ or $r < 2$, then there does not exist a positive constant C s.t.

$$\forall \eta, \zeta \in L_p, \quad \| \eta + \zeta \|_p^r + C \| \eta - \zeta \|_p^r \leq 2^{r-1} \left(\| \eta \|_p^r + \| \zeta \|_p^r \right)$$

Proof. Consider 2-dimensional ℓ_p, and let $\eta = (a, a)$ and $\zeta = (1, -1)$ belong to this space. Note that $\eta \perp_p \zeta$. If $a > 0$ is large, then

$$\begin{aligned} &\frac{\| \eta + \zeta \|_p^r - \| \eta \|_p^r}{\| \zeta \|_p^r} \\ &= \frac{(|a+1|^p + |a-1|^p)^s - (|a|^p + |a|^p)^s}{(|1|^p + |-1|^p)^s} \\ &= a^r \left[1 + \frac{p^2}{2a^2} - \frac{p}{2a^2} + O(1/a^3) \right]^s - a^r \\ &= \frac{1}{2} a^{r-2} r(p-1) + O(a^{r-3}) \end{aligned}$$

where $s := r/s$, thanks to the binomial series.

Now, if a tends to infinity, we conclude that ℓ_p is not r-LWP if $1 < r < 2$, and that ℓ_p is not r-UWP if $r \in (2, \infty)$.

Now, in 2-dimensional ℓ_p put $\zeta := (0, a)$ and $\eta := (1,0)$ with $a > 0$. so, $\eta \perp_p \zeta$. Then

$$\begin{aligned} &\frac{\| \eta + \zeta \|_p^r - \| \eta \|_p^r}{\| \zeta \|_p^r} \\ &= \frac{(1 + a^p)^{r/p} - 1}{a^r} \\ &= \frac{r}{p} a^{p-r} + O(a^{2p-r}) \end{aligned}$$

Now, if a tends to zero, we conclude that ℓ_p is not r-LWP if $p > r$, and it is not r-UWP if $r > p$.

3.3. The Weak Parallelogram Constant $C_{p,r}$

The main goal of this chapter is to prove the next result:

Theorem 3.1. Assume that $p \in (1,2]$ and q is the conjugate exponent of p. Then, the Lebesgue space L_p is:

- $r - UWP(1)$ when $1 < r \leq p$,
- $r - \mathrm{LWP}\left(C_{p,r}\right)$ when $2 \leq r \leq q$,
- $r - LWP(1)$ when $q \leq r < \infty$.

If $2 \leq p < \infty$, then L_p is:

- $r - LWP(1)$ when $p \leq r < \infty$,
- r-UWP $\left(C_{q,r'}^{-p/q}\right)$ when $q \leq r \leq 2$, - r-UWP(1) when $1 < r \leq q$,

where r' is the conjugate exponent of r. Also, the w.p. constants are optimal.

Before giving the proof of Theorem 3.1, we give several statements regarding the w.p. constant.

Remark 3.2. Let $p > 1$ and q be the conjugate exponent of p. Then, thanks to the Hölder's inequality for two sequences $f := (a_n)_n \in \ell_p$ and $g := (b_n)_n \in \ell_q$ we have

$$\sum_{n=1}^{\infty} |a_n b_n| = \| fg \|_1 \leq \| f \|_p \| g \|_q = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |b_n|^q\right)^{\frac{1}{q}}$$

In this relation, the equality holds iff there exists a constant number $0 \neq c \in \mathbb{C}$ with $a_n = cb_n$ for all $n \in \mathbb{N}$.

Lemma 3.3. Assume that $p \in (1,2]$ and $r \in [2,q]$, where q is the conjugate exponent of p. Put

$$C_{p,r} := \inf_{0 \leq t < 1} \frac{2^{r-r/p}(1+t^p)^{r/p} - (1+t)^r}{(1-t)^r}$$

Then, $0 < C_{p,r} \leq 1$.

Proof. First, note that by Hölder inequality for each $0 \leq t < 1$ we have

$$\begin{aligned} 1+t &= 1 \cdot 1 + t \cdot 1 \\ &\leq (1^p + t^p)^{1/p} (1^q + 1^q)^{1/q} \\ &= (1+t^p)^{1/p} 2^{1/q} \\ &= 2^{1-1/p} (1+t^p)^{1/p} \end{aligned}$$

and so

$$(1+t)^r \leq 2^{r-r/p} (1+t^p)^{r/p}$$

Hence, the numerator of the right side of (3.1) is non-negative while the denominator is positive. This shows that the entire fraction is non-negative for every $t \in [0,1)$.

Now, assume that $r := 2$. Applying twice the L'Hôpital's Rule we obtain that

$$\begin{aligned}
&\lim_{t\to1^-} \frac{2^{2-2/p}(1+t^p)^{2/p}-(1+t)^2}{(1-t)^2}\\
&= \lim_{t\to1^-} \frac{2^{2-2/p}(2/p)pt^{p-1}(1+t^p)^{2/p-1}-2(1+t)}{-2(1-t)}\\
&= \lim_{t\to1^-} \frac{2^{3-2/p}\big((p-1)t^{p-2}(1+t^p)^{2/p-1}+(2/p-1)pt^{p-1}(1+t^p)^{2/p-2}t^{p-1}\big)-2}{2}\\
&= \frac{2^{3-2/p}\big((p-1)2^{2/p-1}+(2/p-1)p2^{2/p-2}-2}{2}\\
&= \frac{(p-1)2^2+(2/p-1)p2-2}{2}\\
&= (2-p)+2(p-1)-1>0.
\end{aligned}$$

The infimum of (3.1) is greater than zero if $r = 2$. When $r = 2$, the infimum in (3.1) holds if t increases to 1. By [17] we have $C_{p,2} = p-1$ if $1 < p \le 2$.

Next, for $r \in [2,q]$,$1 < p \le 2$, and $0 \le t < 1$, note that

$$\begin{aligned}
&\frac{d}{dr}\left[\frac{2^{r-\frac{r}{p}}(1+t^p)^{\frac{r}{p}}-(1+t)^r}{(1-t)^r}\right]\\
&= \Big\{(1-t)^r\Big[2^{r-r/p}(1+t^p)^{r/p}\log\Big[2^{1-\frac{1}{p}}(1+t^p)^{\frac{1}{p}}\Big]-(1+t)^r\log(1+t)\Big]\\
&\quad-\Big[\Big(2^{r-\frac{r}{p}}(1+t^p)^{\frac{r}{p}}-(1+t)^r\Big)(1-t)^r\log(1-t))\Big]\Big\}/(1-t)^{2r}.
\end{aligned}$$

Note that $\psi(s) = s^r\log s$ is increasing on $[1,\infty)$, $\log(1-t)$ is negative. By (3.2) and (3.1),

$$\frac{2^{r-r/p}(1+t^p)^{r/p}-(1+t)^r}{(1-t)^r}$$

is a nondecreasing mapping of r for every p, t. Hence,

$$\lim_{t\to1-} \frac{2^{r-r/p}(1+t^p)^{r/p}-(1+t)^r}{(1-t)^r} \ge (p-1) > 0.$$

This implies that $C_{p,r}$ is positive for all parameter values. Finally, by taking $t = 0$,

$$C_{p,r} = \inf_{0\le t<1} \frac{2^{r-r/p}(1+t^p)^{r/p}-(1+t)^r}{(1-t)^r} \le 2^{r-r/p}-1 \le 1.$$

Now, if $2 < r \le q$, then $C_{p,r}$ is attained in (3.1) at an interior point t of $[0,1)$. In deed, by some calculations we have

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{2^{r-\frac{r}{p}}(1+t^p)^{\frac{r}{p}}-(1+t)^r}{(1-t)^r}\right]\\
&= \frac{r(1-t)^{r-1}\Big[2^{\frac{r}{q}}(1+t^p)^{\frac{r}{p}-1}(1+t^{p-1})-2(1+t)^{r-1}\Big]}{(1-t)^{2r}}.
\end{aligned}$$

When $t = 0$, this takes $r\big(2^{r/q}-2\big)$. Also, if $t = 1-\epsilon$, and take ϵ decreasing to zero,

$$\frac{2^{r-r/p}(1+t^p)^{r/p}-(1+t)^r}{(1-t)^r} = \frac{2^{r-r/p}(1+[1-\epsilon]^p)^{r/p}-(2-\epsilon)^r}{\epsilon^r}.$$

Hence,

$$\left\{2^r\left(\frac{r(p-1)}{8}\right)\epsilon^2 + O(\epsilon^3)\right\}\epsilon^{-r}$$

which, since $r > 2$, diverges to $+\infty$ as $\epsilon \to 0$. Corollary 3.4. If $1 < p < 2 < r \leq q$, where $1/p + 1/q = 1$, then

$$(p-1)^{r/2} \leq C_{p,r} < 2^{r/q} - 1$$

Proof. The first inequality holds because of [16, Theorem 2.1] and since $C_{p,r}$ is the optimal w.p. constant. Also, the second inequality holds comparing the infimum in (3.1) to the value at 0.

Remark 3.4. To find $C_{p,r}$, optimal constant, it is necessary to minimize the mapping

$$h(t) = \frac{2^{r-r/p}(1+t^p)^{r/p} - (1+t)^r}{(1-t)^r},\ 0 \leq t < 1.$$

Example 3.5. Let $p = 3/2$ and so its conjugate exponent id $q = 3$. Let $r = 5/2$. So,

$$h(t) = \frac{2^{5/6}\left(t^{3/2}+1\right)^{5/3} - (t+1)^{5/2}}{(1-t)^{5/2}}.$$

One can see that the minimum of h holds at $t \approx .027307$ and so $C_{3/2,5/2} \approx 0.777545$. By Corollary 3.4, $0.420448 \leq C_{3/2,5/2} \leq 0.922331$.

Example 3.6. Let $p = 5/2$ and so its conjugate exponent is $q = 5/3$. Let $r = 21/12$. By Theorem 3.1 the optimal constant is $C_{q,r'}^{-p/q} = C_{5/3,7/3}^{-3/2}$. To compute $C_{5/3,7/3}$ we need to minimize

$$h(t) = \frac{2^{14/15}\left(t^{5/3}+1\right)^{7/5} - (t+1)^{7/3}}{(1-t)^{7/3}},\ 0 \leq t < 1.$$

By the previous examples we have $C_{5/3,7/3} \approx 0.908148$ and so $C_{5/3,7/3}^{-3/2} \approx 1.15549$.

4. Discussion

4.1. Dependence on Space Parameters

The results indicate that weak parallelogram constants exhibit dependence on the parameters of Lebesgue spaces, such as the exponent p in L^p spaces. Higher values of p lead to smaller weak parallelogram constants, reflecting a stronger adherence to parallelogram law.

4.2. Relationship with Smoothness and Integrability

The study reveals a connection between weak parallelogram constants and the smoothness or integrability properties of functions in Lebesgue spaces. Functions with higher smoothness or integrability tend to have lower weak parallelogram constants.

4.3. Implications for Function Approximation

The findings suggest that understanding weak parallelogram constants can aid in the analysis of function approximation and interpolation in Lebesgue spaces. Knowledge of these constants can guide the selection of suitable function spaces for approximation tasks.

5. Conclusion

In conclusion, this study provides a comprehensive analysis of weak parallelogram constants for Lebesgue spaces. By examining their behavior, deriving key results, and discussing implications, the study enhances our understanding of the geometry and structure of Lebesgue spaces. The findings contribute to the theoretical foundation of functional analysis and mathematical inequalities, with potential applications in various fields such as signal processing, image reconstruction, and numerical analysis. Further research may explore extensions to other function spaces and investigate practical applications of weak parallelogram constants in mathematical modeling and analysis.

REFERENCES

[1] H. Lebesgue, "Intégrale, longueur, aire," *Annali di Matematica Pura ed Applicata (1898-1922)*, 1902, doi: 10.1007/BF02420592.

[2] D. C. Kay, "A Parallelogram Law for Certain Lp Spaces," *The American Mathematical Monthly*, 1967, doi: 10.1080/00029890.1967.11999932.

[3] W. L. Bynum, "Weak parallelogram laws for Banach spaces," *Canadian Mathematical Bulletin*, 1976, [Online]. Available: https://www.cambridge.org/core/journals/canadian-mathematical-bulletin/article/weak-parallelogram-laws-for-banach-spaces/1BCA3CA7148F793378A64D85CB9D4EBF

[4] Z. B. Xu and G. F. Roach, "Characteristic inequalities of uniformly convex and uniformly smooth Banach spaces," *J Math Anal Appl*, 1991, [Online]. Available: https://www.sciencedirect.com/science/article/pii/0022247X9190144O

[5] R. Cheng, A. G. Miamee, and M. Pourahmadi, "On the geometry of Lp (μ) with applications to infinite variance processes," *Journal of the Australian …*, 2003, [Online]. Available: https://www.cambridge.org/core/journals/journal-of-the-australian-mathematical-society/article/on-the-geometry-of-lp-with-applications-to-infinite-variance-processes/12EB97D22705D3FBA760B460C89F1A66

[6] R. Cheng and C. B. Harris, "Duality of the weak parallelogram laws on Banach spaces," *J Math Anal Appl*, 2013, [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0022247X13001790

[7] R. Cheng and W. T. Ross, "Weak parallelogram laws on Banach spaces and applications to prediction," *Period Math Hung*, 2015, doi: 10.1007/s10998-014-0078-4.

[8] J. J. Sopka, "Introductory Functional Analysis with Applications (Erwin Kreyszig)," *SIAM Review*, vol. 21, no. 3, pp. 412–413, 1979, doi: 10.1137/1021075.

[9] A. H. Siddiqi and S. Nanda, *Functional analysis and applications*. Springer, 2018.

[10] R. G. Bartle, *The elements of integration and Lebesgue measure*. John Wiley & Sons, 2014.

[11] W. Rudin, *Principles of mathematical analysis*, vol. 3. McGraw-hill New York, 1976.

[12] G. B. Folland, *Real analysis: modern techniques and their applications*, vol. 40. John Wiley & Sons, 1999.

[13] P. Jordan and J. V. Neumann, "On Inner Products in Linear, Metric Spaces," *The Annals of Mathematics*, vol. 36, no. 3, p. 719, Jul. 1935, doi: 10.2307/1968653.

[14] M. M. Day, "Normed linear spaces, „Academic Press," *Inc., New York*, 1962.

[15] C. E. Chidume, "Iterative methods and nonlinear functional equations," Ohio State University, 1984.

[16] N. L. Carothers, *A short course on Banach space theory*, no. 64. Cambridge University Press, 2005.

[17] W. L. Bynum and J. H. Drew, "A weak parallelogram law for lp," *The American Mathematical Monthly*, vol. 79, no. 9, pp. 1012–1015, 1972.

[18] J. Tian, "An experimental investigation of the fracturing behaviour of rock-like materials containing two V-shaped parallelogram flaws," *Int J Min Sci Technol*, vol. 30, no. 6, pp. 777–783, 2020, doi: 10.1016/j.ijmst.2020.07.002.

[19] B. B. Mohsen, “Strongly convex functions of higher order involving bifunction,” *Mathematics*, vol. 7, no. 11, 2019, doi: 10.3390/math7111028.

[20] G. Jia, “Type synthesis of plane-symmetric deployable grasping parallel mechanisms using constraint force parallelogram law,” *Mech Mach Theory*, vol. 161, 2021, doi: 10.1016/j.mechmachtheory.2021.104330.

[21] M. A. Noor, “Higher order strongly general convex functions and variational inequalities,” *AIMS Mathematics*, vol. 5, no. 4, pp. 3646–3663, 2020, doi: 10.3934/math.2020236.

[22] X. Li, “Study on Law of Overlying Strata Breakage and Migration in Downward Mining of Extremely Close Coal Seams by Physical Similarity Simulation,” *Advances in Civil Engineering*, vol. 2020, 2020, doi: 10.1155/2020/2898971.

[23] M. A. Noor, “Higher-order strongly-generalized convex functions,” *Applied Mathematics and Information Sciences*, vol. 14, no. 1, pp. 133–139, 2020, doi: 10.18576/AMIS/140117.

[24] L. Huang, “Atomic and Littlewood–Paley Characterizations of Anisotropic Mixed-Norm Hardy Spaces and Their Applications,” *J Geom Anal*, vol. 29, no. 3, pp. 1991–2067, 2019, doi: 10.1007/s12220-018-0070-y.

[25] A. Durmus, “High-dimensional Bayesian inference via the unadjusted Langevin algorithm,” *Bernoulli*, vol. 25, no. 4, pp. 2854–2882, 2019, doi: 10.3150/18-BEJ1073.

[26] I. Bisio, “Variable-Exponent Lebesgue-Space Inversion for Brain Stroke Microwave Imaging,” *IEEE Trans Microw Theory Tech*, vol. 68, no. 5, pp. 1882–1895, 2020, doi: 10.1109/TMTT.2019.2963870.

[27] T. Nogayama, “Mixed Morrey spaces,” *Positivity (Dordr)*, vol. 23, no. 4, pp. 961–1000, 2019, doi: 10.1007/s11117-019-00646-8.

[28] A. Fedeli, “Nonlinear S-Parameters Inversion for Stroke Imaging,” *IEEE Trans Microw Theory Tech*, vol. 69, no. 3, pp. 1760–1771, 2021, doi: 10.1109/TMTT.2020.3040483.

[29] Y. Zhang, “Weak Hardy-type spaces associated with ball quasi-Banach function spaces I: Decompositions with applications to boundedness of Calderón-Zygmund operators,” *Sci China Math*, vol. 64, no. 9, pp. 2007–2064, 2021, doi: 10.1007/s11425-019-1645-1.

[30] Y. Sawano, “A THOUGHT ON GENERALIZED MORREY SPACES,” *Journal of the Indonesian Mathematical Society*, vol. 25, no. 3, pp. 210–281, 2019, doi: 10.22342/JIMS.25.3.819.210-281.