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Article

Enhancing Numerical Approximations for Scalar Linear Delay Differential Equations with Constant Delay and Impulsive Self-Support Conditions

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Abstract: Through the course of this study, a numerical approximation approach was developed for a category of scalar delay differential equations that were subject to impulsive conditions. Each and every one of these equations was taken into consideration. An innovative method of approximation is used, which involves the utilization of delay equations that have piecewise constant arguments. The theoretical convergence of the approximation technique was shown at each and every point, with the exception of the instants of time that were impulsive. Additionally, the numerical method does a decent job of estimating the instants of time that are impulsive. In the event that the β parameter of the problem is positive, we possess the capability to establish the convergence conclusion by imposing an additional condition via the use of an additional condition. Without making the assumption of a self-supporting condition, this condition guarantees that the solution will cross the line x=c at a temporal interval that is impulsive. The conclusions of the theoretical convergence were shown by the presentation of numerical examples. Through the use of numerical study, we were able to identify the existence of periods in the solution. In this particular solution, the minimal period is defined by being bigger than the delay τ that is associated with the model. We take note of the fact that sufficient conditions are presented to guarantee the existence of periodic solutions; nonetheless, numerical examples illustrate that periodic solutions express themselves under criteria that are considerably less strict than those that guarantee their existence. In light of this, there is a pressing need for more study to be conducted in order to determine whether or not periodic solutions are accessible for this particular set of impulsive equations.

Keywords: delayed differential equations, impulses, equations with piecewise constant arguments, numerical approximation

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1. Introduction

Differential equations that include a delay argument are referred to as delayed differential equations, or DDEs for short [1]. The difference between differential equations and ordinary differential equations is that differential equations take into consideration the function at earlier times in addition to its present value and derivatives [2]. The domains of physics and biology are only two of the many areas in which these equations have found specific applications [3]. Recently, there has been a substantial amount of interest in the process of creating improved numerical approximation techniques for DDEs that include criteria of impulsive self-support [4]. In order to achieve solutions that are more dependable and effective, these strategies intend to enhance the accuracy and convergence features of the techniques that are already in use [5].

For a better understanding of DDEs, let's look at some instances below. An illustration of the notion is provided by the following equations:

First, the equation (1.1.1): The equation $x^{\wedge'}(t) = -2x(t-\pi/2)$

The following equation may be found in equation (2.1.1): $x''(t) = -x'(t) - x'(t-1) - 3\sin(x(t)) + \cos(t)$

$x^{\wedge'}(t)$ equals $x(t)$ minus $x(t/2)$ plus $x^{\wedge'}(t-1)$ according to the equation (3.1.1).

With respect to equation (4.1.1), $x'(t)$ is equal to $x(t)x(t-1)$ plus $t^2x(t+$**2).**

Although (1.1.1) and (2.1.1) are instances of DDEs in this set of equations, (3.1.1) and (4.1.1) are not examples of DDEs since they do not feature delayed terms. Despite the fact that equation (3.1.1) and equation (4.1.1) both have $x^{\wedge'}(t-1)$ on the right side, these equations are not regarded to be DDEs [6].

In order to illustrate how DDEs may be used, let's have a look at the following example:

An example of this would be 1.1.1: Consider the case of a barrel that is filled with brine that has a capacity of B gallons overall. In addition to the water that is being poured to the top of the barrel at a pace of q gallons per minute, the saltwater that is contained inside the barrel is being stirred on a consistent basis. A steady flow of q gallons of seawater per minute is being expelled from the barrel that is located at the bottom of the container [7].

The variable x(t) should be used to represent the amount of salt, measured in pounds of seawater, that is present in the barrel at the moment being signified by the variable. If the saltwater is continuously mixed into the liquid, the remaining saltwater in the barrel will have a concentration of x(t)/B pounds of salt per gallon. This is made on the assumption that the saltwater is always being mixed. Therefore, the equation may be represented in the form that is described below [8].

The expression $x(t) = -q(x(t))/B$

In reality, however, it is quite improbable that the seawater that is contained inside the barrel could be churned up in a single second. Due to the fact that r is a positive constant, the concentration of the saltwater that is still there will be equivalent to the average concentration that was present at some time in the past, which is represented by the symbol t-r. A differential equation for x is turned into a differential differential equation (DDE) in this specific instance [9]: **(5.1.1) the $x'(t)$ equals $-q(x(t-r))$ divided by B**

We may alternatively write the equation as $x^{\wedge'}(t) = -cx(t-r)$ if we define c as q/B. This is another possible formula. On the other hand, this is an alternate method of comprehending the equation. A usual term for this specific kind of DDE is either an R-delay, a time-delay, or a slowness equation. All of these names refer to the same thing [10].

Another distinction between delayed differential equations and ordinary differential equations is that delayed differential equations take into consideration the function's previous values. Ordinary differential equations do not take this into account. Delay arguments are included into the equations in order to create them, and they have a wide range of applications in the domains of physics and biology. It was explained how DDEs may be used in the process of replicating the concentration of saltwater in a barrel over a period of time by the example that was provided [11].

In this inquiry, the emphasis is on scalar linear delay differential equations with constant delay and impulsive self-support conditions. The objective of this investigation is to develop and investigate better numerical approximation approaches for these equations [12]. We wish to compare and contrast the different ways' levels of performance in order to assess the accuracy, stability, and efficacy of the various approaches based on their distinct levels of performance. In addition to this, we will study the convergence features of various techniques and determine the extent to which they are able to be applied to a wide range of problems that present themselves in the real world [13].

2. Materials and Methods

2.1. Problem statement

The simple form of delay differential equations is as follows:

$$y^{'}(t) = ay(t) + by(t-\tau), \quad t \geq t_0$$

$$y(t) = \emptyset(t), \qquad t \leq t_0 (1)$$

where a, b are complex real fixed numbers and τ>0 and ∅ is the given initial function. One of the conditions for the existence of a solution for the delay differential equation is the continuity of ∅.

In this thesis, the linear delay equation with self-supporting impulse conditions is generally as follows:

$$\dot{x}(t) = \alpha x(t) + \beta x(t-\tau) \qquad a.e. t \geq 0 \qquad (2)$$

3. $x(t) = c + d, \quad if\, x(t-) = c$

2.2. Now, considering the initial conditions, we will have:

$$x(t) = \varphi(t), \quad t \in [-\tau,\ 0] \qquad (4)$$

In this case, we will keep in mind throughout the above equation:

$$(H)\ c, d > 0,\ \alpha + |\beta| < 0, \tau > 0,\ c < \varphi(t)\ for\ t \in [-\tau,\ 0],\ \varphi: [-\tau,\ 0] \to R$$

The Lipshitz function is continuous. By solving the impulsive initial value problem (2) and (4) on the function, x is continuous on [0, ∞). There are discontinuities only in time values in relation (2). Otherwise, it looks completely continuous in any interval [0, ∞) that does not contain discontinuity points x. Therefore, in this thesis, we seek to achieve the convergence of the existing method and use numerical results to show Given the approximation convergence, we obtain a numerical approximation scheme for a linear scalar delay differential equation with associated constant delay and with an impulsive self-support condition.

2.3. Delay differential equations with bounded delays and notation, uniqueness and existence

Most delay differential equations have constant delays with delays that are not constant but bounded. Consider the delay differential equation.

$$x'(t) = f(t, x(g_1(t)), \dots, x(g_m(t))(5)$$

We must assume that $t - r \leq g_j(t) \leq t, \forall t \geq t_0 \; and j = 1, 2, \dots, m$ for some constant r≥0

Then the initial function is given as follows.
$x(t) = \emptyset(t), t_0 - r \leq t \leq t_0$

2.4. Note that if r=0, the delay differential equation becomes an ordinary differential equation.
Assume that f is on $[t_0, \beta) \times D^m \to R^n$ for some $\beta > t_0$ and the open set $D \subset R^n$ is defined.

$$F(t, x_t) = f(t, x(g_1(t)), \dots, x(g_m(t))(6)$$

Therefore, equation (5) briefly becomes the following equation.

$$x'(t) = F(t, x_t)(7)$$

2.5. The following definition, which was introduced by Simanov in 1960, is widely used for delay differential equations.

Definition 1 If the function defined on $[t - r, t] \to R^n$, then we define the new function $x_l: [-r, 0] \to R^n$ as follows.

$$x_t(\sigma) = x(t + \sigma) \qquad -r \leq \sigma \leq 0 \qquad (8)$$

Note that x_t is obtained by considering $x(s)$ for $t - r \leq s \leq t$ and transferring this part x to the interval $[-r, 0]$.

2.6. If x is a continuous function, then x_t is a continuous function on $[-r, 0]$.

Notation: We denote the set $C([-r, 0], R^n$ including all continuous functions from $[-r, 0] \to R^n$ by $\mathscr{L}$ and if A is a subset of R^n, we assume do.

$$\mathscr{L}_A = c([-r, 0], A)(9)$$

So if x is continuous on $[t - r' \, t] \to$

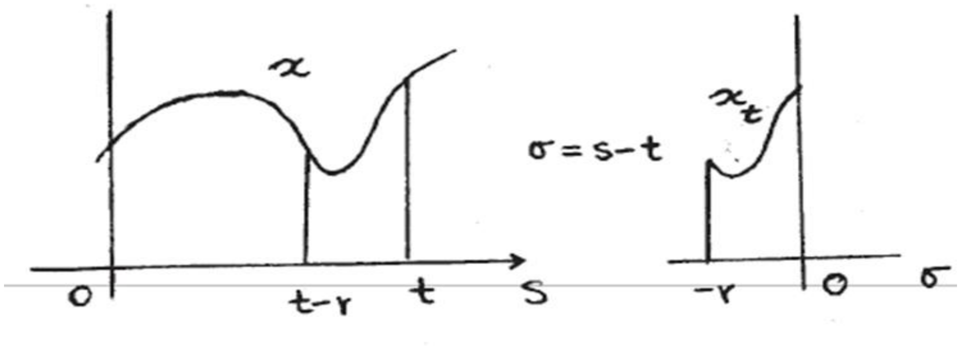
Then: $x_t \in L_A$ Sometimes we use the semi-open interval $[t_0, \beta)$┤ or the full open interval (α, β) so the symbol J for $[t_0, \beta)$ or (α, β) We use delay differential equations with constant coefficients.

2.7. The delay differential equations with constant coefficients will be as follows.

$$x'(t) = \sum_{j=1}^{m} A_j x(t - r_j) + h(t)(10)$$

that A_j are fixed and $0 \leq r_j \leq r$ for j=0, 1, 2,…, m and h(t) is a continuous function on $[t_0, \beta)$. We usually consider equation (6) on $[t_0, \beta)$ with the initial function $x_{t0} = \emptyset$, which $\emptyset \in$L. If h(t)=0, we call the equation homogeneous, so we emphasize more on homogeneous equations.

$$y'(t) = \sum_{j=1}^{m} A_j y(t - r_j) (11)$$



The solutions of equation (15) are

A numerical approximation scheme

In this section, for a class of linear FDEs, we define a simple numerical approximation of IIVP (15)-(16) using EPCAs.

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Prove a discrete parameter h > 0 numerical approximation. The set of mesh points [illegible] $_0$ [illegible] and [illegible] are defined by { [illegible] } [illegible] { [illegible] $_0$ } and { [illegible] }, respectively. We introduce a simplified notation

[illegible])17(

2.8. *The function $t \rightarrow [t]$_h is a piecewise constant function. It has jump discontinuities at points Z h, where it is right-continuous. We associate the following EPCA with self-supporting conditions to IIVP (15)-(16):*

(y_h)˙ (t)= αy_h ([t]_h)+ β_y h ([t]_h −[τ]_h), a.e. t ≥0 (18)
y_h (kh) = c + d, if y_h (kh −) ≤c, (19)
y_h (t)= φ(t), t ∈ [−τ, 0]. (20)

The solution of IIVP (18) – (20) is a function *y_h* : [−τ, ∞) →ℝ which is separable [0 ,∞), and satisfies (18) excluding possible points. N_0 h ; It has a discontinuity point only at the mesh points where (19) exists; and (20) in [-τ,0]. Since the right side of (18) is in fixed intervals [kh,(k + 1)h), every solution of (18) in intervals [kh,(k + 1)h) for k ∈ N. Our opinion is that is that the impulsive condition (19) implies that IIVP (18) - (19) has discontinuities only in some points.

2.9. *Nh. In the following we summarize some properties* $[\cdot]_h$ *we will use later in the article. The definition of* $[t]_h$ *gives the estimate*

$$t - h < [t]_h \leq t,\ t \in R \quad (21)$$

From this rule $[t]_h = t$ uniform in the entire real line of the above estimate implies

$$-h = t - \tau - t + (\tau - h) < t - \tau - ([t]_h - [\tau]_h) < t - \tau - (t - h) + \tau = h$$

Therefore

$$|t - \tau - ([t]_h - [\tau]_h)| < h,\ t \in R. (22)$$

Finally, we mention the relationship

$$[t - jh]_h = [t]_h - jh,\ t \in R,\ j \in Z (23)$$

Fix h > 0 and let $t \in [kh, (k+1)h)$and $l_h := [\tau/h]$. Integrating both sides of (18) from kh to t, and taking the left limit, we get t → (k + 1) h

$$y_h((k+1)h-) = y_h(kh) + h(\alpha y_h(kh) + \beta y_h(kh - l_h h)).(24)$$

We introduce the sequences

$$a(k) := y_h(kh),\ k = -l_h,\ -l_h + 1,\ \cdots,\ and\ b(k) := y_h(kh-),\ k \in N.$$

Then the impulsive self-supporting condition (20) yields a (k) = b(k) if b(k) > c, otherwise a (k) = c + d. Hence, the sequence a(k) and b(k) are obtained.

$$(k+1) = (1 + h\alpha)a(k) + h\beta a(k - l_h),\ k \in N_0,$$
$$a(k+1) = \{b(k+1). \quad b(k+1) > c \ \ c + d. \quad b(k+1) \leq c\ \}(25)$$
$$a(-k) = \varphi(-kh),\ k = 0, 1, \ldots, l_h$$

2.10. *Between kh and (k + 1) h, the function y_h linearly interpolates the values of a (k) and b (k + 1). As a result, we find that IIVP (18) - (20) has a unique solution. Let h > 0 be fixed, and let* $t_0 \in N_0 h,\ \psi \in G$. *We consider the approximate IVP without impulsive terms:*

$$\dot{w}_h(t) = \alpha w_h([t]_h) + \beta w_h([t]_h - [\tau]_h),\ t \geq t_0 \quad (26)$$
$$w_h(t) = \psi(t - t_0),\ t \in [t_0 - \tau, t_0] \quad (27)$$

Introduce the sequence $\underline{a}(k) := w_h(kh + t_0),\ k = -l_h,\ -l_h + 1$ Then, it is easy to see

$$\underline{a}(k+1) = (1 + h\alpha)\underline{a}(k) + h\beta\underline{a}(k - l_h),\ k \in N_0 \quad (28)$$
$$\underline{a}(-k) = \psi(-kh),\ k = 0, 1, \ldots, l_h. \quad (9)$$

3. Results and Discussion

Example: In this example, the initial shock value problem of equation (30) with

X(t) = ax(t) + Bx(t—T), a, t > 0,
x(t) = c + d, if x(t-) = c, (2)
x(t) = g(t), t ∈ [-1, 0].
a = -0.5, f = 0.4, i = 1.0, c = 0.1, d = 0.4, g(t) = 0.2

Consider. Since B > 0, the solution is generally not uniform in time intervals [5 Ent) for n ∈ ℕ. Numerical solution of the impulsive initial value problem

(Int) = ayn (tn) + Byn (tn−1 − [t]n), a0. t ≥ 0
yn(kh) = c + d, if yn(kh−) < c, (31)
Ya (t) = v(t), t ∈ [-τ, 0].

For the values of h=0.1 (blue points) and h=0.001 (red points), we have constructed the figures shown in Figure 1. The picture demonstrates that the numerical approximations for the big step size h=0.1 and for the tiny step size in the h=0.001 time period [0,209] are pretty similar to one another. This is something that can be noticed.

In Figure 1, the horizontal green lines correspond to the levels x=c and x=c+d, the blue dots are the numerical solution for h=0.1 and the purple dots are the numerical solution for h=0.001 in the interval [0,300].

But the numerical solution corresponding to h=0.1 time goes below the level c, 209.4, so due to the impulsive conditions $y_h(t) = \varphi(t), t \in [-\tau, 0]$ creates a fake jump and for the next time 209.4, these two answers differ significantly.

This approximates this type of impulsive problem: if the exact solution is very close to the lower critical value c, a small error in the approximation may cause a similar spurious jump, and then tracking the solution fails.

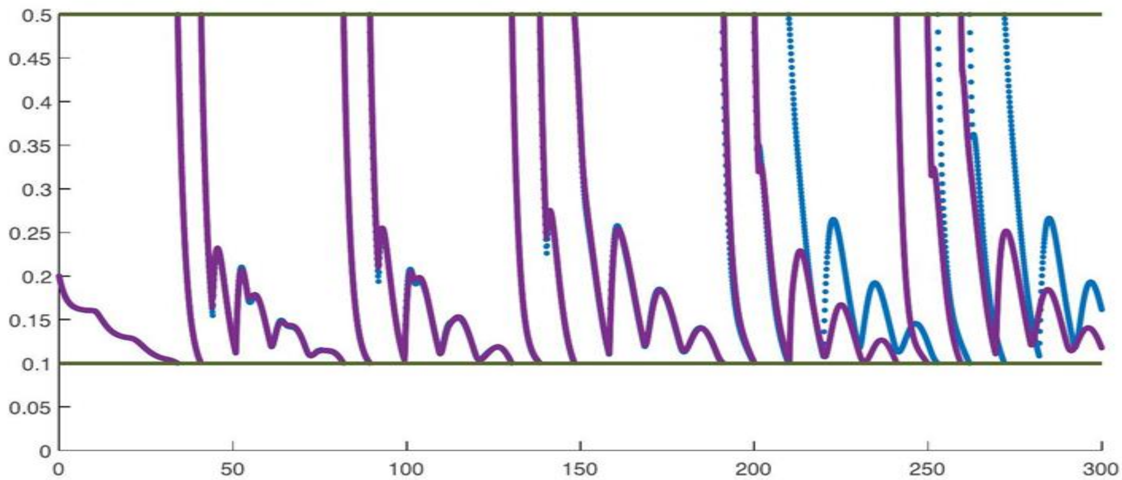

Figure 1 Numerical solution for h=0.1 (blue points) and h=0.001 (red points).

Note that if the numerical solutions for h=0.01, h=0.001 and h=0.0001 even in a long time interval, the resulting graphs match visually, and the convergence of the numerical method is obtained as $h \to 0+$.

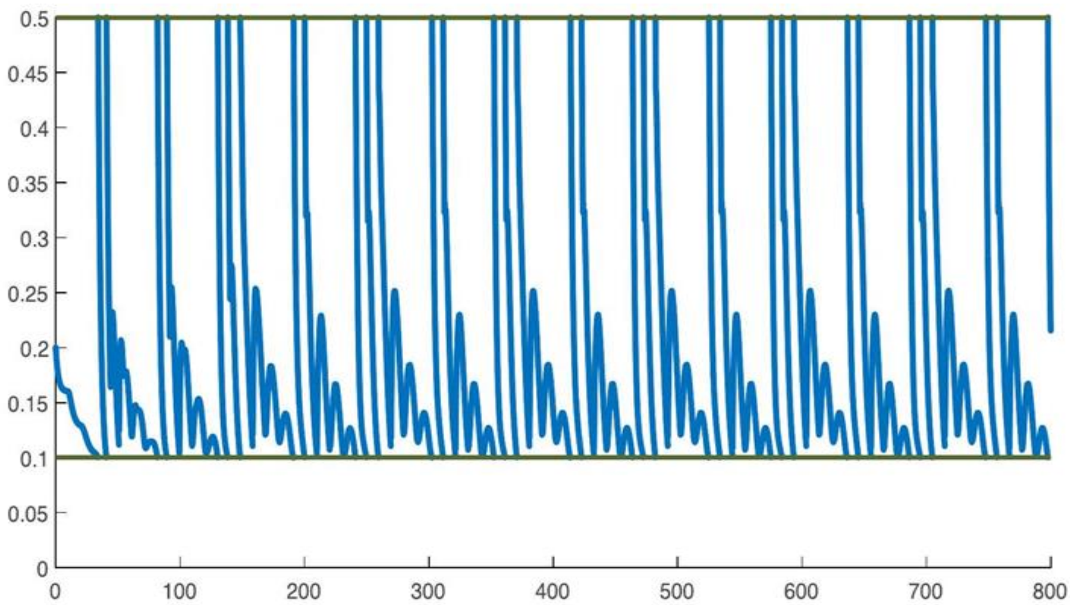

Figure 2 Numerical solution for h=0.001 in [0,800]

Finally, in Figure 2 we present the numerical solution for h=0.001 of the interval [0800]. We observe that the numerical solution is asymptotically periodic, hence the figure indicates the existence of a periodic solution for this example, also with a minimum period around τ>111. Note that there is no known theoretical result for the existence of a periodic solution of the impulsive initial value problem (30) for the case $\beta > 0$.

4. Conclusion

An approach to numerical approximation has been developed for the class of scalar delay differential equations that include impulsive elements. This approach has been given specific attention. As its mathematical foundation, the unique technique of approximation uses delay equations with piecewise constant inputs as its implementation. We were able to show that the approximation scheme is theoretically convergent at every point, with the exception of the impulsive time instants, despite the fact that the numerical approach approximates the impulsive time instants. This was accomplished by demonstrating that the approximation scheme is convergent at every point. In situations when the β parameter of the issue is positive, we possess the capability to exhibit the convergence conclusion by applying an additional constraint. It is the responsibility of this condition to guarantee that the solution crosses the line x=c at a moment that is impulsive, in addition to the self-supporting condition that is expected to be there. To illustrate the theoretical convergence results, we gave numerical examples. This was done in order to illustrate the conclusions. Through the use of numerical research, we were able to demonstrate that there is a periodic solution. This particular solution has a minimum period that surpasses the delay τ that is associated with the model. Despite the fact that this thesis offers sufficient prerequisites to guarantee the existence of periodic solutions, numerical examples illustrate that periodic solutions may exist under conditions that are considerably less robust. Something like this is something that we are aware of. As a result of the fact that the existence of periodic solutions for this set of impulsive equations is something that has to be examined further, extra study is necessary.

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