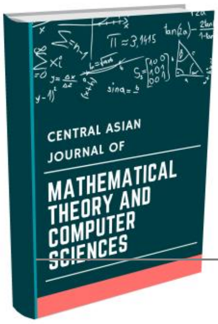

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EMBEDDINGS AND BOUNDEDNESS OF MULTIFUNCTIONAL OPERATORS IN TUBE DOMAINS OVER SYMMETRIC CONES ON BERGMAN AND HILBERT- HARDY SPACES

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Abstract

We show a general sufficient conditions, introduced by Milos Arsenovic and Romi F. Shamoyan[12] for the continuity of the Bergman projection in tube domains over symmetric cones using multifunctional operators embeddings with some sharp embedding relations between the generalized Hilbert-Hardy spaces and the mixed-norm Bergman spaces.

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1-1 Introduction: For $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$of an$n$ - dimensional Euclidean space V. Following [12] we denote the cone Ω by $(1+2\epsilon)$ and by $\Delta$the determinant function on V. For $V = \mathbb{R}^{(3+\epsilon)}$, we have as an example of a symmetric cone on$\mathbb{R}^{(3+\epsilon)}$the Lorentz cone which is a rank 2 cone defined for $\epsilon \geq 0$ by

$$\Lambda_{(3+\epsilon)} = \left\{ y^2 \in \mathbb{R}^{(3+\epsilon)} : y_1^4 - \cdots - y_{(3+\epsilon)}^4 > 0 \right\}.$$

The determinant function in this case is given by the Lorentz form

$$\Delta(y^2) = y_1^2 - \cdots - y_{(3+\epsilon)}^4$$

By using the multi-indices.

If $t = \left((1+2\epsilon)\quad_1, \dots, (1+2\epsilon)_{1+2\epsilon}\right)$,then

$(1+2\epsilon)^{\star} = ((1+2\epsilon)_1, \dots, (1+2\epsilon)_{1+2\epsilon})$and for

$a \in \mathbb{R}, (1+2\epsilon) + a = \left((1+2\epsilon)_1 + a, \dots, (1+2\epsilon)_{(3+\epsilon)} + a\right)$.

Also, if, $(1+2\epsilon), (2+3\epsilon) \in \mathbb{R}^{(3+\epsilon)}$then$\epsilon > 0$ meats $(1+2\epsilon)_{1+\epsilon} < (1+3\epsilon)_{1+\epsilon}$for all $\epsilon \geq 0$.

We use the following multi-index

$$g_0^2 = \left(\frac{2-\varepsilon}{2(1+2\varepsilon)}\right)_{\epsilon\geq 0},$$

For$0 \leq \epsilon \leq \infty$ and $v^2 \in \mathbb{R}^{(3+\epsilon)}$, we denote by $A_{v^2}^{(1+\epsilon,1+2\epsilon)}(T_\Omega)$ the mixed –norm Bergman space consisting of the square analytic functions $f^{\,2}$ in T_Ω such that

$$\|f^2\|_{L_V^{(1+\epsilon,1+2\epsilon)}} = \left(\int_\Omega \left(\int_V |F^2(x^2+iy^2)|^{1+\epsilon} dx^2\right)^{(1+2\epsilon/1+\epsilon)} \Delta_{v^2}(y^2)\frac{dy^2}{\Delta(y^2)^{3+\epsilon/1+2\epsilon}}\right)^{(1/1+2\epsilon)} < \infty$$

where$\Delta_{v^2}$is the generalized power function.

The space $A_{v^2}^{(1+\epsilon,1+2\epsilon)}(T_\Omega)$is nontrivial if and only if $v^2 > g_0^2$, see [6]. When$\epsilon = 0$we write $A_{v^2}^{1+\epsilon,1+2\epsilon}(T_\Omega)$ $=A_{v^2}^{1+\epsilon}(T_\Omega)$ the classical Bergman space $A^{(1+\epsilon)}(\Omega)$correspondsto

$$v^2 = (3+\epsilon/1+2\epsilon, \dots, 3+\epsilon/1+2\epsilon).$$

The (weighted) Bergman projection $P_{v^2}$is the orthogonal projection from the Hilbert space$L_{v^2}^2(T_\Omega)$onto its closed subspace $A_{v^2}^2(T_\Omega)$and it is given by the following integral formula

$$P_{v^2}f^2(z^2) = d_{v^2}\int_{T_\Omega} B_v{}^2(z^2,\omega^2)f^2(\omega^2)dV_{v^2}(\omega^2). \quad (1)$$

where $B_{v^2}(z^2,\omega^2) = c_{v^2}\Delta^{-\left(v^2+\frac{3+\epsilon}{1+2\epsilon}\right)}((z^2-\varpi^2)/i)$ is the Bergman reproducing kernel for$A_{v^2}^2$. Hence we used $dV_{v^2}(\omega^2) = \Delta^{v^2-\frac{3+\epsilon}{1+2\epsilon}}du^2dv^2$, where $\omega^2 = u^2 + iv^2 \in T_\Omega$.

The problem of boundedness of the Bergman projection on tube domains over symmetric cones has been considered (see [12]). The best known results is given in [9] in the setting of the light cone. Recently, an equivalent condition for the boundedness of the Bergman projection in terms of Hardy-type inequalities and duality was shown in [12]. We introduce (see [12]) here the operators $T_\beta, \beta = \left(\beta_1, \dots, \beta_{(1+\epsilon)}\right)$which generalize the Bergman projection i.e.

$$T_\beta\left(\overrightarrow{f^2}\right)\left(\overrightarrow{z^2}\right) = \int_{T_\Omega} \frac{\left(\prod_\epsilon^{(1+\epsilon)} f_{1+\epsilon}^2\, z^2\right)\Delta^{\frac{1}{(1+\epsilon)}}\sum_\epsilon^{(1+\epsilon)}\beta_{1+\epsilon}(\Im z^2)}{\prod_\epsilon^{(1+\epsilon)}\Delta^{\frac{1}{(1+\epsilon)}\left(\frac{3+\epsilon}{1+2\epsilon}+\beta_{1+\epsilon}\right)}\Delta^{\frac{3+\epsilon}{1+3\epsilon}}(\Im z^2)}$$

where $\overrightarrow{f^2} = \left(f_1^2, \dots, f_{(3+\epsilon)}^2\right)$, $\overrightarrow{z^2} = \left(z_1^2, \dots, z_{(3+\epsilon)}^2\right), z_{1+\epsilon}^2 \in T_\Omega$ and $f_{1+\epsilon}^2 \in L_{loc}^1(T_\Omega)$

$for\ \epsilon \geq 0$. Combining classical function we obtain the following sufficient condition for the boundedness of operator T_β from the product space

$$\prod_{t+\epsilon}^{(1+\epsilon)} L^{1+\epsilon}_{(1+\epsilon)v^2_{(1+\epsilon)}+\frac{3\epsilon+\epsilon^2}{1+\epsilon}}(T_\Omega \quad) = L^{1+\epsilon}_{(1+\epsilon)v_1^2(2+\epsilon)\frac{3+\epsilon}{1+\epsilon}}(T_\Omega \quad) \times \dots \times L^{1+\epsilon}_{(1+\epsilon)v_1^2+\frac{3\epsilon+\epsilon^2}{1+\epsilon}}(T_\Omega \quad)$$

to the space $L^{1+\epsilon}\left((T_\Omega)^{(1+\epsilon)}, \prod_{\epsilon}^{(1+\epsilon)} \Delta^{v^2_{(1+\epsilon)}} dV(z^2_{1+\epsilon})\right)$. We consider such multifunctional operator see[10]. Some results here are analogous to results of [10] proven in the case of the unit ball in $\mathbb{C}^{(3+\epsilon)}$. It is suare that almost all multifunctional results given here are well known in the case $\epsilon = 0$. For example, the case $\epsilon = 0$ of the following theorem is in [4].

Theorem (1-1): Let$v^2_{(1+\epsilon)} \in \mathbb{R}$, $\epsilon = 0, \dots, 1+\epsilon, 1 \le \epsilon \le \infty$ and $\beta = (\beta_1, \dots, \beta_{3+\epsilon})$

If the parameters satisfy the following conditions

$$\frac{1}{1+\epsilon}\sum_{\epsilon=0}^{1+\epsilon} \beta_{1+\epsilon} > \frac{\epsilon-2}{3+\epsilon} \qquad (2)$$

Embedding relations and boundedness.

$$0 \le \epsilon < 2+\epsilon\left(\frac{(3+\epsilon)min_{(1+\epsilon)}v^2_{(1+\epsilon)} - \epsilon + 2}{\epsilon - 2}\right) \qquad (3)$$

$$min_{1+\epsilon}\beta_{1+\epsilon} > \frac{1}{1+\epsilon}\sum_{\epsilon=0}^{1+\epsilon}\beta_{1+\epsilon} - \frac{3+\epsilon}{(3+\epsilon)_{1+\epsilon}} + 1\left(\frac{2-\epsilon}{1+2\epsilon} + max_{(1+\epsilon)}v^2_{(1+\epsilon)}\right) \quad (4)$$

then T_β is bounded from

$$\prod_{\epsilon=0}^{1+\varepsilon} L^{1+\epsilon}_{(1+\epsilon)v^2_{(1+\epsilon)}+\frac{3\epsilon+\epsilon^2}{1+\epsilon}}(T_\Omega) \text{ to } L^{1+\epsilon}\left((T_\Omega)^{(1+\epsilon)}, \prod_{1+4\epsilon}^{(1+\epsilon)} \Delta^{v^2_{(1+\epsilon)}-\frac{3+\epsilon}{1+2\epsilon}} dV(z^2_{1+3\epsilon})\right).$$

Through the applications of the above result, we obtain a sufficient condition of the boundedness of the Bergman projection in terms of the reproducing formula, which is new (see [12]).

Theorem (1-2): Let $v^2 > \frac{3+\epsilon}{1+2\epsilon}$ and $0 < \epsilon < \infty$. If for any $f^2 \in L^{1+\epsilon}_{v^2}(T_\Omega)$ the following representation formula holds

$$P_{v^2}f^2(z_1^2)P_{v^2}f^2(z_2^2) = C_\beta \int_{T_\Omega} \frac{f^2(z^2)P_{v^2}f^2(z^2)\Delta^{\beta-\frac{3+\epsilon}{1+2\epsilon}}(\Im z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}+\frac{\beta}{2}\right)}\left(\frac{z_1^2-\bar{z}^2}{i}\right)\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}+\frac{\beta}{2}\right)}\left(\frac{z_2^2-\bar{z}^2}{i}\right)} dV^2(z^2) \quad (5)$$

for some sufficiently large β and all z_1^2, z_2^2in T_Ω, then the Bergman projection P_{v^2} is bounded on $L^{1+\epsilon}_{v^2}(T_\Omega)$.

In the above theorem the weights v^2 and β are taken real, but the result generalizes directly to the vector weight case. The condition " β is sufficiently large "is related to the boundedness conditions for the Bergman kernel and determinant function. For example, a necessary condition for the boundedness of the Bergman projection P_{v^2}on $L^{1+\epsilon}_{v^2}(T_\Omega)$ is that related Bergman kernel belongs to $L^{(1+\epsilon)',(1+2\epsilon)'}_{v^2}(T_\Omega)$ where $\epsilon = 0$and this can only happen for large values of β for $(1+\epsilon), (1+2\epsilon)$ and v^2 fixed, see [11].

We considered the embedding relation between some generalization of the classical Hardy spaces and the weighted mixed norm Bergman spaces in the tube domains over general cones (see [12]). Let $\mathcal{H}^{1+\epsilon}(T_\Omega)$denotes the holomorphic Hardy space on the tube domain, i.e. the space of holomorphic square functions f^2on$T_\Omega$such that

$$\|f^2\|_{\mathcal{H}^{1+\epsilon}} = \left(sup_{(1+2\epsilon)\in\Omega}\int_{\mathbb{R}^{3+\epsilon}} \left|f^2(x^2 + i(1+2\epsilon))\right|^{1+\epsilon} dx^2\right)^{1/1+\epsilon} < \infty$$

Following [8], we extend the above definition of Hardy spaces to a more general family of spaces $\mathcal{H}_{\mu}^{1+\epsilon}(T_\Omega)$ for any locally finite and quasi-invariant measure μ supported on $\bar{\Omega}$. The space $\mathcal{H}_{\mu}^{1+\epsilon}(T_\Omega)$ consists of all square functions f^2 holomorphic in T_Ω satisfying

$$\|f^2\|_{\mathcal{H}_{\mu}^{1+\epsilon}(T_\Omega)} = \left(sup_{(1+2\epsilon)\in\Omega} \int_{T_\Omega} \left|f^2\left(x^2 + i\left(y^2 + (1+2\epsilon)\right)\right)\right|^{1+\epsilon} dx^2 d\mu(y^2)\right)^{1/1+\epsilon} < \infty$$

In particular, if $\mu = \delta_0$, this space coincides with the classical Hardy space and if μ is the Lebesgue measure, it coincides with the Bergman space $A^{1+\epsilon}(T_\Omega)$.

We consider only those measures μ which are obtained by analytic continuation from the family $d\mu_s$of measures

$$d\mu_{(1+2\epsilon)}(1+2\epsilon) = \chi_\Omega(1+2\epsilon)\frac{\Delta_{(1+2\epsilon)}(1+2\epsilon)}{\Gamma_\Omega(1+2\epsilon)}\frac{dt}{\Delta^{3+\epsilon/1+2\epsilon}}, (1+2\epsilon) \in \mathbb{R}^{(3+\epsilon)},$$

$$(1+2\epsilon) > g_0^2,$$

where $\Gamma_\Omega$denotes the gamma function of the cone Ω.

In the family $\{\mu_{(1+2\epsilon)}\}_{(1+2\epsilon)\in\mathbb{C}^{(1+2\epsilon)}}$ of tempered distributions we consider only those which are positive measures. These measures come from a characterization of Gindikin and correspond to those $(1+2\epsilon) = \left((1+2\epsilon)_1, \dots, (1+2\epsilon)_{(3+\epsilon)}\right) \in \mathbb{C}^{1+2\epsilon}$ which belong to the following Wallach set

$$\Xi = \left\{\left(u_1^2, u_1^2 + \frac{d}{2}sgnu_1^2, \dots, u_{1+2\epsilon}^2 + \frac{d}{2}(sgnu_1^2 + \cdots + sgnu_1^2)\right): u_1^2, \dots\ u_1^2 \geq 0\right\}.$$

We are interested in the embedding relations between $\mathcal{H}_{\mu}^{(1+\epsilon)}(T_\Omega)$ and $A_{v^2}^{(1+\epsilon,1+2\epsilon)}(T_\Omega)$,

$\mu = \mu_{(1+2\epsilon)}, (1+2\epsilon) \in\ \Xi$and $v^2 \in \mathbb{R}^{1+2\epsilon}$. We show the following sharp result (see [12]).

Theorem (1-3): Let$(1+2\epsilon) \in\ \Xi, \mu = \mu_{(1+2\epsilon)}, v^2 \in \mathbb{R}^{1+2\epsilon}$ and, $v^2 > g_0^2$. Then for $0 \leq \epsilon < \infty$ with $\frac{q^2}{2}(1+2\epsilon) > g_0^2$we have: $\mathcal{H}_{\mu}^2(T_\Omega)\ \hookrightarrow\ A^{2+2\epsilon}_{\frac{u^2}{2}.(1+2\epsilon)}(T_\Omega)$if and only if

$$\frac{4+4\epsilon+4\epsilon^2}{2+4\epsilon} = \frac{v^2}{1+2\epsilon} + \left(\frac{3+\epsilon}{1+\epsilon}, \dots, \frac{v^2}{1+3\epsilon+2\epsilon^2}\right).$$

We note that for the sufficiency part of the above theorem it suffices to prove that

$\mathcal{H}_{\mu}^{1+\epsilon}(T_\Omega)\ \hookrightarrow\ A^{2.u^2}_{\frac{u^2}{2}.1+2\epsilon}(T_\Omega)$for all $u^2 \geq 2$ such that $\frac{u^2}{2}(1+2\epsilon) > g_0^2$. This is a consequence of the embedding relations between Bergman spaces. The condition

$\frac{u^2}{2}(1+2\epsilon) > g_0^2$ shows that for$(1+2\epsilon)$ fixed, u^2should be sufficiently large and so this theorem is not applicable in all cases. It is clear that the usual Hardy space $\mathcal{H}^2$ is not covered by this theorem.

Theorem (1-4): Let$(1+2\epsilon) \in \Xi, \mu = \mu_{(1+2\epsilon)}, v^2 \in R^{1+2\epsilon}\ and\ v^2 > g_0^2$. Then for $0 \leq \epsilon < \infty$ we have:

$\mathcal{H}_{\mu}^2(T_\Omega)\ \hookrightarrow\ A_{v^2}^{(1+\epsilon,1+2\epsilon)}(T_\Omega)$ if and only if

$$\left(\frac{3+\epsilon}{4+8\epsilon}, \dots, \frac{3+\epsilon}{4+8\epsilon}\right) + \frac{1}{4}\left(\frac{8+10\epsilon+8\epsilon^2}{4+8\epsilon}\right) = \frac{v^2}{1+2} + \left(\frac{3+\epsilon}{1+3\epsilon+2\epsilon^2}, \dots, \frac{3+\epsilon}{1+3\epsilon+2\epsilon^2}\right).$$

Then the sufficiency part of this theorem can be reduced to the proof of the embedding $\mathcal{H}_{\mu}^2(T_\Omega)\ \hookrightarrow$ $A^{4,u^2}_{\frac{u^2}{4}\left(\frac{5+9\epsilon+8\epsilon^2}{1+2\epsilon}\right)}(T_\Omega)$for $u^2 \geq 4$.

The necessity parts of Theorems (1-3) and (1-4) follow exactly as in Proposition 2.25 of [6] with the use of norm identity provided in Proposition 3.1 of [8] for $\mathcal{H}^2_\mu(T_\Omega)$. In order to prove sufficiency, we heavily rely on Paley – Wiener theory in this setting. The only difference with the one –dimensional case is that one has to deal with the beta function of the tube domain with respect to the rotated Jordan frame. That is, one needs a version of Theorem VII. 1.7. of [7] where the generalized determinant function is replaced by the one corresponding to the rotated Jordan frame (see [5]).

Let c denote positive constants, not necessarily the same at different occurrences on parameters is indicated by subscripts. Given two quantities A and B the notation $A \lesssim B$. When both $A \lesssim B$and $B \lesssim A$ hold we write $A \approx B$.

2 Preliminaries and Auxiliary Results:

2.1 Symmetric Cones and the Generalized Determinant Function:

ForΩ be an irreducible open cone of $(1+2\epsilon)$rank in $(3+\epsilon)-dimensional$vector space endowed with an inner product $(./.)$ for which Ω is self –dual. Let $G(\Omega)$ be the group of transformations of Ω and G its identity component. There is a subgroup $H$of G acting simply transitively on Ω, i.e. for every $y^2 \in \Omega$there is a unique $g^2 \in$ Hsuch that $y^2 = g^2\boldsymbol{e}$, where **e** is fixed element inΩ.

Note that Ω induces in V a structure of Euclidean Jordan algebra with identity **e** such that

$$\overline{\Omega} = \{x^4 : x^2 \in V\}$$

We can identify(since Ω is irreducible)the inner product with the one given by the trace on V:

$$(x^2/y^2) = tr\,(x^2y^2), x^2, y^2 \in V.$$

For $\{c_1, \dots, c_{1+2\epsilon}\}$ be a fixed Jordan frame in V and

$$V = \bigoplus_{\epsilon\geq\infty} V_{i,1+\epsilon}$$

be its associated Pierce decomposition of V. We denote by $\Delta_1\,(x^2), \dots, \Delta_{1+2\epsilon}\,(x^2)$ the principal minors of $x^2 \in V$ with respect to the fixed Jordan frame $\{c_1, \dots, c_{1+2\epsilon}\}$. Hence, $\Delta_{(1+3\epsilon)}\,(x^2)$ is the determinant of the projection $P_k x^2$ of x^2 in the Jordan sub algebra:

$$V^{(1+3\epsilon)} = \bigoplus_{\epsilon\geq\infty} V_{i,1+\epsilon}$$

We have Δ=$\Delta_{(1+2\epsilon)}$and $\Delta_{(1+3\epsilon)}\,(x^2) > 0, \epsilon \geq 0$ when $x^2 \in \Omega$. The generalized power function on Ωdefined as

$$\Delta_{(1+2\epsilon)}\,(x^2) = \Delta_1^{(1+2\epsilon)_1-(1+2\epsilon)_2}\,(x^2)\,\Delta_2^{(1+2\epsilon)_2-(1+2\epsilon)_3}\,(x^2)\cdot\Delta_{1+2\epsilon}^{(1+2\epsilon)_{1+2\epsilon}}\,(x^2),$$
$$x^2 \in \Omega, (1+2\epsilon) \in \mathbb{C}^{1+2\epsilon}$$

We have the generalized gamma function associated to Ω:

$$\Gamma_\Omega(1+2\epsilon) = \int_\Omega \begin{matrix} \boldsymbol{e}^{-(e/\xi^2)}\,\Delta_{(1+2\epsilon)}\,(\xi^2)\,\Delta^{-3+\epsilon/1+2\epsilon}\,(\xi^2)d\xi^2, \\ (1+2\epsilon) = \left((1+2\epsilon)_1, \dots, (1+2\epsilon)_{(1+2\epsilon)}\right) \in \mathbb{C}^{1+2\epsilon} \end{matrix}$$

This integral converges if and only if $\Re(1+2\epsilon)_{(1+\epsilon)} > \frac{3+\epsilon}{2} = (1+\epsilon)\frac{d}{2}$for all $\epsilon \geq 0$

In that case we have a formula:

$$\Gamma_\Omega(1+2\epsilon) = (2\pi)^{\frac{2-\epsilon}{2}} \prod_{1+\epsilon}^{1+2\epsilon} \Gamma\left((1+\epsilon)_{(1+\epsilon)} - \frac{2-\epsilon}{2(1+2\epsilon)}\right)$$

We have the following result on the Laplace transform of the generalized power function

Lemma (2.1): Let$(1+\epsilon)=((1+\epsilon)_1,\ldots,(1+\epsilon)_{1+2\epsilon})\in\mathbb{C}^{3+\epsilon}$ with $\Re(1+\epsilon)_{(1+\epsilon)}>\frac{2-\epsilon}{2(1+2\epsilon)}, \varepsilon=1,\ldots,(1+2\epsilon)$. Then, for all $y^2\in\Omega$we have

$$\int_\Omega -i^{(y^2/\xi^2)}\Delta_{(1+2\epsilon)}(\xi^2)\Delta^{-(3+\epsilon)/1+2\epsilon}(\xi^2)d\xi^2=\Gamma_\Omega(1+2\epsilon)\left[\Delta^*_{(1+2\epsilon)^*}(y^2)\right]^{-1}$$

Here $y^2=h^2e$ if and only if $y^{-2}=h^{2(*-1)}\boldsymbol{e}$with $h^2\in H$ *and* $\Delta^*_{(1+\epsilon)}, \epsilon=1,\ldots,1+2\epsilon$ arethe principal minors with respect to the rotated Jordan frame$\{c_1,\ldots,c_{1+2\epsilon}\}$.

The beta function of the symmetric cone Ω is defined by the following integral

$$B_{\Omega(1+\epsilon,1+2\epsilon)}=\int_{\Omega\cap(e-\Omega)}\Delta_{\frac{2\epsilon^2+2\epsilon-2}{1+2\epsilon}}(x^2)\Delta_{\frac{4\epsilon^2+3\epsilon-2}{1+2\epsilon}}(e-x^2)dx^2$$

where $(1+\epsilon)$, $(1+2\epsilon)$ $\epsilon\mathbb{C}^{1+2\epsilon}$. When $\Re(1+\epsilon)_j, \Re(1+2\epsilon)_{(1+\epsilon)}>\frac{2-\epsilon}{2(1+2\epsilon)}$ the above integral converges absolutely and

$$B_{\Omega(1+\epsilon,1+2\epsilon)}=\frac{\Gamma_{(1+\epsilon)}\Gamma_{(1+2\epsilon)}}{\Gamma_\Omega(2+3\epsilon)}$$

(see Theorem VII.1.7 in [7]).

Let $m$be an element of $G_{\boldsymbol{e}}$, the stabilizer of $\boldsymbol{e}$ in Gsuch that

$$(1+\epsilon)c_{(1+\epsilon)}=c_{(\epsilon+1)}, \qquad (1+\epsilon)=1,\ldots,(1+2\epsilon)$$

Then for any $y^2\in\Omega$ and $(1+2\epsilon)\in\mathbb{C}^{1+2\epsilon}$,$\Delta^*_{(1+2\epsilon)}(y^2)=\Delta_{(1+2\epsilon)}((1+\varepsilon)^{-1}y^2)$identity, oneobtainsas in the proof of Theorem VII.1.7. of [7] the following result (see [5] for details).

Lemma (2.2): Let$y^2\in\Omega$. The integral

$$F^2(y^2)=\int_{(y^2-\Omega)\cap\Omega}\Delta^*_{(1+\epsilon)^*-\frac{3+\epsilon}{1+2\epsilon}}(x^2)\Delta^*_{(1+2\epsilon)^*-\frac{3+\epsilon}{1+2\epsilon}}(y^2-x^2)dx^2$$

converges if $\Re_{(1+\epsilon)_{(1+\epsilon)}}, \Re_{(1+2\epsilon)_{(1+\epsilon)}}>\frac{2-\epsilon}{2(1+2\epsilon)}$ *for* $(1+\epsilon)=1,\ldots,(1+2\epsilon)$. In this case.

2.2 Bergman Spaces with Kernel Function Integrability:

We show some estimates for the function in the Bergman space or the projections of the functions in $L^{1+\epsilon,1+2\epsilon}_{v^2}(T_\Omega)$. We begin with a pointwise estimate of elements in $A^{1+\epsilon,1+2\epsilon}_{v^2}(T_\Omega)$. The following lemma follows from the invariance of the Bergman spaces with respect to the transformation group (G_Ω) (see [6]).

Lemma (2.3): Let $0\le\epsilon<\infty$ and $v^2\in\mathbb{R}^r, v^2>g_0^2$. Then

$$|f^2(z^2)|\lesssim \Delta-\frac{v^2}{1+2\epsilon}-\frac{3+\epsilon}{1+3\epsilon+\epsilon^2}(\Re z^2)\|f^2\|_{A^{(1+\epsilon,1+2\epsilon)}_{v^2}}, \qquad z^2\in T_\Omega$$

We also need a pointwise estimate for the Bergman projection of functions in $L^{1+\epsilon,1+2\epsilon}(T_\Omega)$, defined by integral formula (1), when this projection makes sense. We state the following integrability properties for the determinant function.

Lemma (2.4): Let $\alpha\in\mathbb{C}^{(1+2\epsilon)}$ and$y^2\in\Omega$.

1) The integral

$$J_0(y^2)=\int_{\mathbb{R}^{(3+\epsilon)}}\left|\Delta_{-\alpha}\left(\frac{x^2+iy^2}{i}\right)\right|dx^2$$

converges if and only if$\boldsymbol{\Re\alpha}>g_0^2+\frac{\mathbf{3+\epsilon}}{\mathbf{1+\epsilon}}$. In that case$J_0(y^2)=C_\alpha\left|\Delta_{-\alpha+(3+\epsilon)/(1+\epsilon)}(y^2)\right|$.

2) For any multi-indices$(1+2\epsilon)$ and β and $(1+2\epsilon)\in\Omega$the function $y^2\longmapsto\Delta_\beta(y^2+1)\;\Delta_{(1+2\epsilon)}\,(y^2)$belongs to $L^1\left(\Omega,\frac{dy^2}{\Delta^{3+\epsilon,1+2\epsilon}\,(y^2)}\right)$if and only if $\boldsymbol{\Re\alpha}>g_0^2$and $\Re(\alpha+\beta)<g_0^{2*}$. In thatcase we have

$$\int_{\mathbb{R}^{(3+\epsilon)}}\Delta_\beta(y^2+1+2\epsilon)\Delta_{(1+2\epsilon)}(y^2)\frac{dy^2}{\Gamma^{(3+\epsilon)/(1+2\epsilon)}(y^2)}=C_{\beta,(1+2\epsilon)}\Delta_{(1+2\epsilon)+\beta}(1+2\epsilon)$$

Of [6] for the proof of the above lemma. Let τ denotes the set of all triples $(1+\epsilon,1+\epsilon,v^2)$ such that $\epsilon\geq 0$and the function$B_{v^2}(\cdot,\boldsymbol{ie})$belongs to$L^{(1+\epsilon)'}_{(1+2\epsilon)'}(T_\Omega)$. We have the following pointwise estimate (see [12]).

Lemma (2.5): Suppose $(1+\epsilon,1+\epsilon,v^2)\;\in\tau$. Then

$$|P_{v^2}(f^2z^2)|\leq\Delta_{-\frac{v^2}{1+2\epsilon}-\frac{(3+\epsilon)}{(1+2\epsilon)(1+\epsilon)}}(z^2)\|f^2\|_{L^{(1+\epsilon,1+2\epsilon)}_{v^2}}$$

Proof: This is an easy consequence of the above lemma and $H\ddot{o}lder$ inequality.

We conclude this section with a useful embedding relation between mixed norm Bergman spaces (see [6] for an alternative proof) and [7].

Lemma (2.6): Suppose $0\leq\epsilon<\infty$and$v^2,\beta>g_0^2$. Then $A^{(1+2\epsilon,1+2\epsilon)}_{v^2}(T_\Omega)\hookrightarrow A^{(1+2\epsilon,1+2\epsilon)}_{\beta}(T_\Omega)$ if and only if $\frac{v^2(1+3\epsilon)+3+\epsilon}{(1+3\epsilon+2\epsilon^2)}=\frac{\beta(1+2\epsilon)+3+\epsilon}{(1+4\epsilon+2\epsilon^2)}$.

Proof: Let us suppose that $\frac{v^2(1+3\epsilon)+3+\epsilon}{(1+3\epsilon+2\epsilon^2)}=\frac{\beta(1+2\epsilon)+3+\epsilon}{(1+4\epsilon+2\epsilon^2)}$. We recall that there is a sequence of points $\left(x^2{}_{(1+\epsilon,1+3\epsilon)}+y^2(1+3\epsilon)\right)_{(1+\epsilon,1+3\epsilon)\in Z^2}$such that

$$\|f^2\|^{(1+\epsilon)}_{A^{l,(1+\epsilon)}_\mu}\approx\sum_{1+3\epsilon}\left(\sum_{1+2\epsilon}\left|f^2\left(z^2_{(1+2\epsilon,1+3\epsilon)}\right)\right|^l\right)^{(1+\epsilon)/l}\Delta_{\mu+\frac{(3+\epsilon)}{(1+2\epsilon)}\frac{(1+\epsilon)}{l}}\left(y^2(1+3\epsilon)\right)$$

see [5] and [4]. From this and embeddings between $l^{(1+\epsilon)}$ spaces we obtain

$$\|f^2\|^{(1+2\epsilon)}_{A^{(1+2\epsilon,1+2\epsilon)}_\mu}\approx\sum_{1+3\epsilon}\left(\sum_{1+2\epsilon}\left|f^2\left(z^2_{(1+2\epsilon,1+3\epsilon)}\right)\right|^{(1+2\epsilon)}\right)^{(1+\epsilon)/l}\Delta_{\beta+\frac{(3+\epsilon)\,(1+2\epsilon)}{(1+2\epsilon)(1+2\epsilon)}}\left(y^2(1+3\epsilon)\right)$$

$$\leq\left(\sum_{(1+3\epsilon)}\left(\sum_{(1+\epsilon)}\left|f^2\left(z^2_{(1+\epsilon,1+2\epsilon)}\right)\right|^{(1+\epsilon)}\right)^{(1+2\epsilon)/(1+\epsilon)}\Delta_{\frac{(3+\epsilon)}{(1+\epsilon)}}\left(y^2(1+3\epsilon)\right)\right)\approx\|f^2\|^{(1+2\epsilon)}_{A^{(1+\epsilon,1+2\epsilon)}_{v^2}}$$

For the converse, we test with functions $B_\mu(\cdot,x^2+iy^2)$ where μ is large enough and is fixed in x^2+iy^2. Now continuity of the embedding and Lemma (2.4) give

$$\Delta_{-(1+2\epsilon)\mu+\beta\frac{1}{(1+2\epsilon)}+\frac{(3+7\epsilon+2\epsilon^2)}{(1+3\epsilon+2\epsilon^2)}}(y^2)\leq c\Delta_{-(1+2\epsilon)\mu+v^2\frac{1}{(1+2\epsilon)}+\frac{(3+7\epsilon+2\epsilon^2)}{(1+3\epsilon+2\epsilon^2)}}(y^2),\;y^2\in\Omega$$

which implies that $\frac{v^2(1+\epsilon)+(3+\epsilon)}{(1+3\epsilon+2\epsilon^2)}=\frac{\beta(1+2\epsilon)+(3+\epsilon)}{(1+2\epsilon)^2}$.

As a first application of the above to prove that $\mathcal{H}^2_\mu\,(T_\Omega)\hookrightarrow A^{2,u^2}_{\frac{u^2}{2}(1+2\epsilon)}(T_\Omega)$for all $u^2\geq 2$such that $\frac{u^2}{2}>g_0^2$. In fact, if $(1+\epsilon,1+2\epsilon)$ and v^2 satisfy the hypotheses of Theorem (1.3) then, by the above lemma, we have

$A^{2,u^2}_{\frac{u^2}{2}}(T_\Omega) \hookrightarrow A^{(1+\epsilon,1+2\epsilon)}_{v^2}(T_\Omega)$ with $u^2 \le (1+2\epsilon)$. Similarly, for proof of sufficiency in Theorem 1.4 it suffices to prove that $\mathcal{H}^2_\mu\,(T_\Omega) \hookrightarrow A^{4,u^2}_{\frac{u^2}{4}(4+5\epsilon)}(T_\Omega)$ for all $u^2 \ge 4$.

3.1 Multifunctional Bergman –Type Operators and Embeddings:

We denote by $\boxdot = \Delta\left(\frac{1}{i}\frac{\partial}{\partial x^2}\right)$ the partial differential operator of $(1+2\epsilon)$ on $\mathbb{R}^{(3+\epsilon)}$defined by

$$\boxdot\left[e^{i(x^2|\xi^2)}\right] = \Delta(\xi^2)e^{i(x^2|\xi^2)},\ \ x^2, \xi^2 \in \mathbb{R}^{(3+\epsilon)}.$$

3.1 MultifunctionalBergman –type Operators:

We investigate boundedness of the operator $T_\beta$from $\prod_{3\epsilon\ge0}^{(1+\epsilon)} L^{(1+\epsilon)}_{(1+\epsilon)v^2_{(1+3\epsilon)}}(T_\Omega)$ to $L^{(1+\epsilon)}\left((T_\Omega)\prod_{\epsilon\ge0}^{(1+\epsilon)}\Delta^{v^2_{(1+3\epsilon)}-\frac{(3+\epsilon)}{(1+2\epsilon)}}dV\left(z^2_{(1+3\epsilon)}\right)\right)$. We apply the obtained result to multifunctional embeddings for functions in the Bergman spaces$A^{(1+\epsilon)}_{v^2}(T_\Omega)$where $v^2 > \frac{2-\epsilon}{(1+2\epsilon)}$and $\epsilon \ge 0$. We begin with the following result (see [12]). .

Theorem (1.5): Let $v^2 = \left(v_1^2, \dots, v^2_{(1+\epsilon)}\right) \in \mathbb{R}^{(1+\epsilon)}, \epsilon > 0$and $\epsilon \ge 0, \beta = \left(\beta_1, \dots, \beta_{(1+\epsilon)}\right) \in \mathbb{R}^{1+\varepsilon}$. If the parameters satisfy the following conditions

$$\frac{1}{(1+\epsilon)}\sum_{\epsilon\ge0}^{(1+\epsilon)}\beta_{(1+\epsilon)} > \frac{(2-\epsilon)}{(1+2\epsilon)}$$

$$0 \le \epsilon < 2 + \epsilon\left(\frac{\min_{(1+\epsilon)}v^2_{(1+\epsilon)}}{(2-\epsilon)}(1+2\epsilon) - 1\right), \text{ and}$$

$$\min_{(1+\epsilon)}\beta_{(1+\epsilon)}\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+\epsilon)} - \frac{(3+\epsilon)}{(1+3\epsilon+2\epsilon^2)} + \frac{(1+\epsilon)}{(1+\epsilon)}\left(\frac{(4-2\epsilon)}{(1+2\epsilon)} - 1 + \max_{(1+\epsilon)}v^2_{(1+\epsilon)}\right)$$

then $T_\beta$is bounded from $\prod_{\epsilon\ge0}^{(1+\epsilon)} L^{(1+\epsilon)}_{(1+\epsilon)v^2_{(1+\epsilon)}+(\epsilon)\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(T_\Omega)$ to $L^{(1+\epsilon)}\left((T_\Omega)^{(1+\epsilon)}, \prod_{\epsilon=0}^{(1+\epsilon)}\Delta^{v^2_{(1+\epsilon)}-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}dV\left(z^2_{(1+3\epsilon)}\right)\right)$. The ideas of proof is taken from [10].

Proof: Using *Hölder* inequality we obtain

$$\left|T_\beta\left(\vec{f}^2\left(z^2_{(1+\epsilon)}, \dots, z^2_{(1+\epsilon)}\right)\right)\right|^{(1+\epsilon)} = \left|\int_{T_\Omega}\frac{\left(\prod_{\epsilon=0}^{(1+\epsilon)} f^2_{\epsilon=0}(z^2)\right)\Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+\epsilon)}}(\Im z^2)}{\prod_{\epsilon=0}^{(1+\epsilon)}\Delta^{\frac{1}{(1+\epsilon)}\left(\left(\frac{3+\epsilon}{1+2\epsilon}+\beta_{(1+\epsilon)}\right)\right)}\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)}\frac{dV(z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)}\right|^{(1+\epsilon)} \ge I \times J$$

where

$$I = \int_{T_\Omega}\frac{\left(\prod_{\epsilon=0}^{(1+\epsilon)}|f^2_{\epsilon=0}(z^2)|^{(1+\epsilon)}\right)\Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+\epsilon)}}(\Im z^2)}{\prod_{\epsilon=0}^{(1+\varepsilon)}\left|\Delta\left(\frac{z^2-\bar{z}^2}{i}\right)\right|^{(1+\epsilon)\alpha_{(1+\epsilon)}}}\frac{dV(z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)}$$

$$J^{(1+\epsilon)'/(1+\epsilon)} = \int_{T_\Omega}\frac{\left(\prod_{\epsilon=0}^{(1+\epsilon)}|f^2_{\epsilon=0}(z^2)|^{(1+\epsilon)}\right)\Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+\epsilon)}}(\Im z^2)}{\prod_{\epsilon=0}^{(1+\epsilon)}\left|\Delta\left(\frac{z^2-\bar{z}^2}{i}\right)\right|^{(1+\epsilon)'\gamma_{(1+\epsilon)}}}\frac{dV(z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)}$$

and $\alpha_{(1+\epsilon)} + \gamma_{(1+\epsilon)} = \frac{1}{(1+\epsilon)}\left(\frac{3+\epsilon}{1+2\epsilon} + \beta_{(1+\epsilon)}\right)$

Let us choose$\gamma_{(1+\epsilon)}$such that $\gamma_{(1+\epsilon)} > \frac{1}{(1+\epsilon)}\left(\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)} + \frac{5}{1+2\epsilon}\right)$.Then we estimate the integral J using $H\ddot{o}lder$ inequality and Lemma (1.4)

$$J^{(1+\epsilon)'/(1+\epsilon)} = \int_{T_\Omega} \prod_{\epsilon=0}^{(1+\epsilon)} \left|\Delta\left(\frac{z^2 - \bar{z}^2}{i}\right)\right|^{(1+\epsilon)'\gamma_{(1+\epsilon)}} \Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)} - \left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)\, dV\,(z^2)$$

$$\geq (1+\epsilon)\prod_{\epsilon=0}^{(1+\epsilon)}\left(\int_{T_\Omega}\left|\Delta\left(\frac{z^2 - \bar{z}^2}{i}\right)\right|^{(1+\epsilon)(1+\epsilon)'\gamma_{(1+\epsilon)}} \Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)} - \left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)\, dV\,(z^2)\right)^{1/(1+\epsilon)}$$

$$= (1+\epsilon)\prod_{\epsilon\geq 0}^{(1+\epsilon)} \Delta^{-(1+\epsilon)'\gamma_{(1+\epsilon)} + \frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)} - \left(\frac{3+\epsilon}{1+2\epsilon}\right)}\left(\Im z^2_{(1+\epsilon)}\right)$$

Hence we obtained

$$J \leq (1+\epsilon)\prod_{\epsilon\geq 0}^{(1+\epsilon)} \Delta^{-(1+\epsilon)\gamma_{(1+\epsilon)} + \frac{1}{(1+\epsilon)(1+\epsilon)'}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)} + \frac{(1+\epsilon)}{(1+\epsilon)'}\left(\frac{3+\epsilon}{1+2\epsilon}\right)}\left(\Im z^2_{(1+\epsilon)}\right)$$

Using the estimate (13) and Lemma (2.4) we finally obtain

$$\int_{T_\Omega} \dots \prod_{\epsilon=0}^{(1+\epsilon)} \left|T_\beta(\vec{f}^2)(z_1^2, \dots z^2_{(1+\epsilon)})\right|^{(1+\epsilon)} \Delta^{v^2_{(1+3\epsilon)} - \frac{3+\epsilon}{1+2\epsilon}}\, dV(z_1^2) \dots dV\left(z^2_{(1+\epsilon)}\right)$$

$$(1+\epsilon)\int_{T_\Omega}\left(\prod_{\epsilon=0}^{(1+\epsilon)} \left|f^2_{(1+\epsilon)}(z^2)\right|^{(1+\epsilon)}\right) g^2(z^2)\, \Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)}}(\Im z^2)\, dV \frac{dV(z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)}$$

where

$$\int_{T_\Omega} \dots \int_{T_\Omega} \prod_{\epsilon=0}^{(1+\epsilon)} \left(\begin{array}{c} \left|\Delta\left(\frac{z^2_{(1+3\epsilon)} - \bar{z}}{i}\right)\right|^{-(1+\epsilon)\alpha_{(1+3\epsilon)}} \\ \Delta^{v^2_{(1+3\epsilon)} - \frac{3+\epsilon}{1+2\epsilon} - (1+\epsilon)\gamma_{(1+\epsilon)} + \frac{1}{(1+\epsilon)(1+\epsilon)'}\sum_{\epsilon=0}^{(1+3)} \beta_{(1+\epsilon)} + \frac{(3+4\epsilon+\epsilon^2)}{(1+3\epsilon+\epsilon^2)(1+\epsilon)'}}\left(\Im z_{(1+3\epsilon)}\right) \end{array}\right)$$
$$dV(z_1^2) \dots dV\left(z^2_{(1+\epsilon)}\right)$$

Note that (11) implies $(1+\epsilon)\alpha_{(1+3\epsilon)} > v^2_{(1+3\epsilon)} - (1+\epsilon)\gamma_{(1+3\epsilon)} + \frac{1}{(1+\epsilon)(1+\epsilon)'}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)} + \frac{(3+4\epsilon+\epsilon^2)}{(1+3\epsilon+\epsilon^2)(1+\epsilon)'} + \frac{5}{(1+2\epsilon)}$. Then, if we finally choose$\alpha_{(1+\epsilon)}$and $\gamma_{(1+\epsilon)}$such that (12) holds and, for every $\epsilon = 0, \dots, 1+\varepsilon$, we have

$$\frac{1}{(1+\epsilon)(1+\epsilon)'}\left(\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)} \beta_{(1+\epsilon)} + \frac{5}{(1+2\epsilon)}\right) < \gamma_{(1+\epsilon)}$$

$$< min\left\{\begin{array}{c}\frac{1}{(1+\epsilon)}\left(\frac{3+\epsilon}{1+2\epsilon}+\beta_{(1+\epsilon)}\right),\frac{min_{(1+\epsilon)}v^2_{(1+3\epsilon)}+\frac{(3+\epsilon)}{(1+\epsilon)}+1}{(1+\epsilon)} \\ +\frac{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+\epsilon)}+\frac{(3+\epsilon)}{(1+2\epsilon)}}{(1+\epsilon)(1+\epsilon)'}\end{array}\right\}$$

then an application of Lemma (2.4) gives estimate

$$g^2(z^2)\leq(1+\epsilon)\Delta^{\sum_{\epsilon=0}^{(1+\epsilon)}v^2_{(1+3\epsilon)}+(1+\epsilon)\frac{(3+\epsilon)}{(1+2\epsilon)}-(1+\epsilon)\sum_{\epsilon=0}^{(1+\epsilon)}(\alpha_{(1+3\epsilon)}+\beta_{(1+3\epsilon)})+\frac{1}{(1+\epsilon)'}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+3\epsilon)}+\frac{(1+\epsilon)}{(1+2\epsilon)(1+\epsilon)'}}(\Im z^2)$$

Finally, using $Hölder$ inequality

$$\int_{T_\Omega}\dots\int_{T_\Omega}\prod_{\epsilon=0}^{(1+\epsilon)}\left|T_\beta\left(\overrightarrow{f^2}\right)\left(z_1^2,\dots z^2_{(1+\epsilon)}\right)\right|^{(1+\epsilon)}\Delta^{v^2_{(1+3\epsilon)}-\frac{(3+\epsilon)}{(1+2\epsilon)}}\left(\Im z^2_{(1+3\epsilon)}\right)dV(z_1^2)\dots dV^2\left(z^2_{(1+\epsilon)}\right)$$

$$\leq(1+\epsilon)\int_{T_\Omega}\left(\prod_{\epsilon=0}^{(1+\epsilon)}\left|f^2_{(1+\epsilon)}(z^2)\right|^{(1+\epsilon)}\right)\Delta^{\sum_{\epsilon=0}^{(1+\epsilon)}v^2_{(1+3\epsilon)}+(\epsilon)\frac{(3+\epsilon)}{(1+2\epsilon)}}\frac{dV(z_1^2)}{\Delta^{\frac{(3+\epsilon)}{(1+2\epsilon)}}}\leq$$

$$(1+\epsilon)\left(\prod_{\epsilon=0}^{(1+\epsilon)}\int_{T_\Omega}\left|f^2_{(1+\epsilon)}(z^2)\right|^{(1+\epsilon)}\Delta^{(1+\epsilon)v^2_{(1+3\epsilon)}+\frac{(3\epsilon+\epsilon^2)}{(1+2\epsilon)}}\frac{dV(z_1^2)}{\Delta^{\frac{(3+\epsilon)}{(1+2\epsilon)}}}\right)<\infty.$$

An analogue of the following lemma in the setting of the unit ball in $\mathbb{C}^{(3+\epsilon)}$ is contained in [10]. Note that the case $\epsilon>0$is obvious (see [12]).

Lemma (2.7): Let$v^2_{(1+3\epsilon)}>\frac{2-\epsilon}{(1+2\epsilon)}$, $\epsilon=0,\dots,1+\epsilon$ and $\epsilon\geq 0$. Then there is a constant $\epsilon>0$ such that

$$\int_{T_\Omega}\prod_{\epsilon=0}^{(1+\epsilon)}\left|f^2_{(1+3\epsilon)}(z^2)\right|^{(1+\epsilon)}\Delta^{(\epsilon)\frac{(3+\epsilon)}{(1+2\epsilon)}+\sum_{\epsilon=0}^{\epsilon}v^2_{(1+3\epsilon)}-\frac{(3+\epsilon)}{(1+2\epsilon)}}(\Im z^2)dV(z^2)$$

$$\leq C(1+\epsilon)\prod_{\epsilon=0}^{(1+\epsilon)}\left\|f^2_{(1+3\epsilon)}\right\|^{(1+\epsilon)}_{A^{(1+\epsilon)}_{v^2_{(1+3\epsilon)}}}$$

Proof: By Lemma (2.6) we have $A_{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}v^2_{(1+3\epsilon)}}(T_\Omega)\hookrightarrow A^{(1+\epsilon)}_{(\epsilon)\frac{(3+\epsilon)}{(1+2\epsilon)}+\sum_{\epsilon=0}^{(1+\epsilon)}v^2_{(1+3\epsilon)}}(T_\Omega)$ to prove the lemma, we only need to check that for$f^2_{(1+\epsilon)}\in A^{(1+\epsilon)}_{v^2_{(1+3\epsilon)}}(T_\Omega),\epsilon\geq 0$ the product$f_1^2\dots f^2_{(1+\epsilon)}$ is in $A_{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}v^2_{(1+3\epsilon)}}(T_\Omega)$ with the appropriate norm estimate. Anapplication of $Hölder$ inequality

$$\int_{T_\Omega}\prod_{\epsilon=0}^{(1+\epsilon)}\left|f^2_{(1+3\epsilon)}(z^2)\right|^{(1+\epsilon)}\Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}v^2_{(1+3\epsilon)}-\frac{(3+\epsilon)}{(1+2\epsilon)}}(\Im z^2)dV(z^2)$$

$$\leq\prod_{\epsilon=0}^{(1+\epsilon)}\left(\int_{T_\Omega}\left|f^2_{(1+3\epsilon)}\right|^{(1+\epsilon)}\Delta^{v^2_{(1+3\epsilon)}-\frac{(3+\epsilon)}{(1+2\epsilon)}}(\Im z^2)dV(z^2)\right)^{1/(1+\epsilon)}$$

finishes the proof since the last expression is equal to $\prod_{\epsilon=0}^{(1+\epsilon)}\left\|f^2_{(1+3\epsilon)}\right\|^{(1+3\epsilon)/(1+\epsilon)}_{A^{(1+\epsilon)}_{v^2_{(1+3\epsilon)}}}$.

A complete analogue of the following multifunctional result in the setting of the until ball can be found in [10] (see [12]).

Theorem (1.6): Let$v^2_{(1+3\epsilon)} > \frac{(3+\epsilon)}{(1+2\epsilon)}$for $\epsilon \geq 0$, let $0 \leq \epsilon < \infty$ and suppose that $\beta_{(1+\epsilon)}$ are sufficiently large so that for any sequence$\left(z^2_{(1+\epsilon)}\right)_{\epsilon=0}^{1+\varepsilon}$ in T_Ω the following representation holds for $f_1^2, \ldots, f_{1+\varepsilon}^2 \in \mathcal{H}(T_\Omega)$

$$f_1^2(z_1^2)\ \ldots, f_{1+\varepsilon}^2(z_{1+\varepsilon}^2) = (1+\epsilon)_{1+\varepsilon,\beta} \int_{T_\Omega} \frac{\prod_{\epsilon=0}^{1+\varepsilon} \boldsymbol{f}^{\mathbf{2}}_{(\mathbf{1}+\boldsymbol{\epsilon})}(\boldsymbol{z}^{\mathbf{2}}) \Delta^{\frac{1}{1+\varepsilon}\sum_{\epsilon=0}^{1+\varepsilon}\boldsymbol{\beta}_{(1+\epsilon)}}(\mathfrak{J}\boldsymbol{z}^{\mathbf{2}})}{\prod_{\epsilon=0}^{1+\varepsilon} \Delta^{\frac{1}{1+\varepsilon}\left(\frac{(3+\epsilon)}{(1+2\epsilon)}+\beta_{(1+\epsilon)}\right)}\left(\frac{z^2_{(1+\epsilon)} - \bar{z}^2}{i}\right)}$$

Assuming none of the functions$\boldsymbol{f}^{\mathbf{2}}_{(\mathbf{1}+\mathbf{3}\boldsymbol{\epsilon})}$ if identically zero, the following statements are equivalent.

1) There is a constant $\epsilon > 0$ such that

$$\int_{T_\Omega} \prod_{\epsilon=0}^{(1+\epsilon)} \left|f^2_{(1+\epsilon)}\right|^{(1+\epsilon)} \Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+\epsilon)}}(\mathfrak{J}z^2) \frac{dV(z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\mathfrak{J}z^2)}(1+\epsilon) < \infty$$

2) $f^2_{(1+3\epsilon)} \in A^{(1+\epsilon)}_{v^2_{(1+3\epsilon)}}(T_\Omega)\ for\ all\ \epsilon = 0, \ldots, (1+\epsilon).$

Proof: We havealready seen that 2) ⇒ 1) independently of the representation formula (15).

Let us prove implication 1) ⇒ 2) assuming (15). Since the functions are not identically zero, condition

$$\int_{T_\Omega} \cdots \int_{T_\Omega} \prod_{\epsilon=0}^{(1+\epsilon)} \left\|f^2_{(1+3\epsilon)}\left(z^2_{(1+3\epsilon)}\right)\right\|^{(1+\epsilon)} \Delta^{v^2_{(1+3\epsilon)} - \frac{(3+\epsilon)}{(1+2\epsilon)}}\left(z^2_{(1+3\epsilon)}\right) dV\left(z^2_{(1+3\epsilon)}\right) \ldots dV\left(z^2_{(1+3\epsilon)}\right) < \infty$$

implies $f^2_{(1+3\epsilon)} \in A^{(1+\epsilon)}_{v^2_{(1+3\epsilon)}}(T_\Omega)\ for\ all\ \epsilon = 0, \ldots, 1+\varepsilon$, Now, using the representation (15) we obtain

$$k = \int_{T_\Omega} \cdots \int_{T_\Omega} \prod_{\epsilon=0}^{(1+\epsilon)} \left(\overrightarrow{f^2}\left|f^2_{(1+3\epsilon)}\left(z^2_{(1+3\epsilon)}\right)\right|^{(1+\epsilon)} \Delta^{v^2_{(1+3\epsilon)} - \frac{(3+\epsilon)}{(1+2\epsilon)}}\left(z^2_{(1+3\epsilon)}\right)\right) dV\left(z^2_{(1+3\epsilon)}\right) \ldots dV\left(z^2_{(1+3\epsilon)}\right)$$

$$= \int_{T_\Omega} \cdots \int_{T_\Omega} \left|T_\Omega\left(\overrightarrow{f^2}\left(z_1^2, \ldots, z^2_{(1+\epsilon)}\right)\right|^{(1+\epsilon)} \left(\prod_{\epsilon=0}^{(1+\epsilon)} \Delta^{v^2_{(1+3\epsilon)} - \frac{(3+\epsilon)}{(1+2\epsilon)}}(\mathfrak{J}z^2)\right) dV\left(z^2_{(1+3\epsilon)}\right) \ldots dV\left(z^2_{(1+3\epsilon)}\right)$$

where $\overrightarrow{f^2} = \left(f_1^2, \ldots, f^2_{(1+\epsilon)}\right)$. The proof of Theorem (1.5) gives.

$$\int_{T_\Omega} \cdots \int_{T_\Omega} \prod_{\epsilon=0}^{(1+\epsilon)} \left(\left\|f^2_{(1+3\epsilon)}\left(z^2_{(1+3\epsilon)}\right)\right\|^{(1+\epsilon)} \Delta^{v^2_{(1+3\epsilon)} - \frac{(3+\epsilon)}{(1+2\epsilon)}}\left(\mathfrak{J}z^2_{(1+3\epsilon)}\right) dV\left(z^2_{(1+3\epsilon)}\right) \ldots dV\left(z^2_{(1+3\epsilon)}\right) < \infty$$

We write $\left(v^2, (1+\epsilon)\right) \in \sigma\ if\ v^2 \in \mathbb{R}, \epsilon \geq 0,\ v^2 > \frac{2-\epsilon}{1+2\epsilon}\Delta^{-\left(v^2+\frac{3+\epsilon}{1+2\epsilon}\right)}\left(\frac{z^2-ie}{i}\right) \in L^{(1+\epsilon)'}_{v^2}$.

Let us define, for$f^2_{(1+3\epsilon)} \in L^{(1+\epsilon)}_{v^2_{(1+3\epsilon)}}$, the following operations:

$$S_{\beta,(1+3\epsilon)}\left(\overrightarrow{f^2}\right)\left(\overrightarrow{z^2}\right) = \int_{T_\Omega} \frac{f^2_{(1+3\epsilon)}(z^2) \prod_{\epsilon=3\epsilon} P_{v^2_{(1+\epsilon)}} f^2_{(1+\epsilon)}(z^2)\, \Delta^{\frac{1}{(1+\epsilon)}\sum_{\epsilon=0}^{(1+\epsilon)}\beta_{(1+\epsilon)}}(\mathfrak{J}z^2)}{\prod_{\epsilon=0}^{(1+\epsilon)} \Delta^{v^2_{(1+3\epsilon)} - \left(\frac{3+\epsilon}{1+2\epsilon}\right)}\left(\frac{z^2_{(1+\epsilon)} - \bar{z}^2}{i}\right)} \frac{dV(z_1^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\mathfrak{J}z^2)}$$

and

$$(1+2\epsilon)_\beta = \sum_{\epsilon=0}^{(1+\epsilon)} (1+2\epsilon)_{\beta,(1+3\epsilon)}$$

We show the following (see [12]),

Theorem (1.7): Suppose$\left(v_{(1+3\epsilon)}^2, (1+\epsilon)\right) \in \sigma$ if the parameters satisfy conditions (2), (3) and (4), then the operators $S_{\beta_{(1+3\epsilon)}}$and $S_\beta$are bounded from

$$\prod_{\epsilon=0}^{(1+\epsilon)} L_{v_{(1+\epsilon)}^2}^{(1+\epsilon)}(T_\Omega)\ to\ \left(L^{(1+\epsilon)}(T_\Omega)^{(1+\epsilon)}, \prod_{\epsilon=0}^{(1+\epsilon)} \Delta^{v_{(1+\epsilon)}^2-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)dV(z_1^2)\right)$$

Proof: Clearly we only need to prove the result for $S_{\beta_{(1+3\epsilon)}}$fixed$(1+3\epsilon)$. An inspection of the proof of Theorem (1.5)and Lemma (2.5) give

$$\int_{T_\Omega}\int_{T_\Omega} \left|S_{\beta,(1+3\epsilon)}\left(\overrightarrow{f^2}\left(\overrightarrow{z^2}\right|^{(1+\epsilon)} \Delta^{v_{(1+\epsilon)}^2-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)dV(z_1^2) \dots dV\left(z_{(1+\epsilon)}^2\right) \le$$

$$(1+\epsilon)\int_{T_\Omega} \left|f_{(1+3\epsilon)}^2(z^2)\right|^{(1+\epsilon)} \left(\prod_{\epsilon\neq 3\epsilon}^{(1+\epsilon)} \left|P_{v_{(1+\epsilon)}^2} f_{(1+\epsilon)}^2(z^2)\right|^{(1+\epsilon)}\right) \Delta^{\sum_{\epsilon=0}^{(1+\epsilon)} v_{(1+\epsilon)}^2+(\epsilon)\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)\frac{dV^2(z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)}$$

$$(1+\epsilon) \le \prod_{\epsilon\neq 3\epsilon} \left\|f_{(1+\epsilon)}^2\right\|_{L_{v_{(1+\epsilon)}^2}^{(1+\epsilon)}} \int_{T_\Omega} \left|f_{(1+3\epsilon)}^2(z^2)\right|^{(1+\epsilon)} \Delta^{v_{(1+\epsilon)}^2-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)dV(z^2) \le (1+\epsilon)\prod_{\epsilon\neq 3\epsilon}\left\|v_{(1+\epsilon)}^2\right\|_{L_{v_{(1+\epsilon)}^2}^{(1+\epsilon)}}$$

and the proof is complete.

Theorem (1.8): Suppose$v_{(1+\epsilon)}^2 \in \sigma\ for\ \epsilon = 0, \dots, (1+\epsilon)$ Suppose also that, for $\beta_{(1+\epsilon)}$large enough, the following representation

$$\prod_{\epsilon=0}^{(1+\epsilon)} P_{v_{(1+3\epsilon)}^2} f_{(1+3\epsilon)}^2\left(z_{(1+3\epsilon)}^2\right) = (1+\epsilon)_{(1+\epsilon),\beta}\int_{T_\Omega} \frac{f_{(1+3\epsilon)}^2 \prod_{j\neq k}^{(1+\epsilon)} P_{v_{(1+3\epsilon)}^2} f^2(z^2)\Delta^{\frac{1}{2}\sum_{\epsilon=0}^{m}\beta_{(1+\epsilon)}}(\Im z^2)}{\Delta^{\frac{1}{2}\left(\frac{(3+\epsilon)}{(1+2\epsilon)}+\frac{\beta}{2}\right)}\left(\frac{z^2-\bar{z}^2}{i}\right)\Delta^{\frac{1}{2}\left(\frac{(3+\epsilon)}{(1+2\epsilon)}+\frac{\beta}{2}\right)}\left(\frac{z^2-\bar{z}^2}{i}\right)}\frac{dV(z^2)}{\Delta^{\left(\frac{3+\epsilon}{1+2\epsilon}\right)}}$$

holds for any sequence $\left(z_{(1+\epsilon)}^2\right)_{\epsilon=0}^{1+\varepsilon}$ in $T_\Omega$and any $f_{(1+3\epsilon)}^2 \in L_{v^2}^{(1+\epsilon)}(T_\Omega), \epsilon \ge 0$.

We also have the following corollary which gives a sufficient condition for boundedness of the Bergman projection (see [12]).

Corollary (1.1): Let$\left(v^2, (1+\epsilon)\right) \in \sigma$**.** If the following representation

$$P_{v^2} f^2(z_1^2) P_{v^2} f^2(z_1^2) = (1+\epsilon)_\beta \int_{T_\Omega} \frac{f^2(z^2)P_{v^2}f^2(z^2)}{\Delta^{\frac{1}{2}\left(\frac{(3+\epsilon)}{(1+2\epsilon)}+\frac{\beta}{2}\right)}\left(\frac{z^2-\bar{z}^2}{i}\right)\Delta^{\frac{1}{2}\left(\frac{(3+\epsilon)}{(1+2\epsilon)}+\frac{\beta}{2}\right)}\left(\frac{z^2-\bar{z}^2}{i}\right)} \Delta^{\beta-\frac{(3+\epsilon)}{(1+2\epsilon)}}(\Im z^2)dV(z^2)$$

holds for all $z_1^2, z_2^2 \in T_\Omega$ and $f^2 \in L_{v^2}^{(1+\epsilon)} T_\Omega$, where β is large enough, thenP_{v^2} is bounded on$L_{v^2}^{(1+\epsilon)}(T_\Omega)$.

Proof: Using Lemma(2.5) we clearly have

$$\int_{T_\Omega} |f^2(z^2)|^{(1+\epsilon)} |P_{v^2} f^2(z^2)|^{(1+\epsilon)} \Delta^{2v^2}(\Im z^2)dV(z_1^2)$$

$$\le (1+\epsilon)\|f^2\|_{L_{v^2}^{(1+\epsilon)}}^{(1+\epsilon)} dV(z^2)\int_{T_\Omega} |f^2(z^2)|^{(1+\epsilon)} \Delta^{v^2-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z^2)dV(z^2) = (1+\epsilon)\|f^2\|_{L_{v^2}^{(1+\epsilon)}}^{2(1+\epsilon)}$$

Now, following the proof of Theorem (1.7) we obtain

$$\|P_{v^2}f^2\|^{2(1+\epsilon)}_{L^{(1+\epsilon)}_{v^2}} = \int_{T_\Omega}\int_{T_\Omega} |P_{v^2}f^2(z^2)|^{(1+\epsilon)}|P_{v^2}f^2(z_2^2)|^{(1+\epsilon)}\Delta^{v^2-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z_1^2)\Delta^{v^2-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(\Im z_2^2)\, dV(z_1^2)dV(z_2^2)$$

$$\leq (1+\epsilon)\int_{T_\Omega} |f^2(z^2)|^{(1+\epsilon)}\, |P_{v^2}f^2(z_2^2)|^{(1+\epsilon)}\Delta^{2v^2}(\Im z^2)\, dV(z^2)$$

$$\leq (1+\epsilon)\int_{T_\Omega} \|f^2\|^{(1+\epsilon)}_{L^{(1+\epsilon)}_{v^2}}$$

3.2 Multifunctional Inequalities Involving Bergman Projection or the Box Operator:

We derive multifunctional inequalities involving the Bergman projection or the box operator. As a preparation (see [12]).

Proposition (1.1): Let $(v^2, (1+\epsilon)) \in \sigma$ If $P_{v^2}$is bounded on $L^{(1+\epsilon)}_{v^2}(T_\Omega)$then$P_{v^2}$is bounded, from$L^{(1+\epsilon)}_{v^2}(T_\Omega)\ to\ L^{(1+4\epsilon+\epsilon^2)}_{(1+3\epsilon)v^2+3\epsilon\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(T_\Omega)\ for\ any\ (1+3\epsilon)\in\mathbb{N}$

. **Proof:** SupposeP_{v^2} is bounded on $L^{(1+\epsilon)}_{v^2}$. Then using Lemma (2.5) we obtain, fro any $f^2 \in L^{(1+\epsilon)}_{v^2}(T_\Omega)$:

$$\int_{T_\Omega} |P_{v^2}f^2(z_2^2)|^{(1+\epsilon)}\Delta^{v^2}(\Im z^2)\, dV(z^2) = \int_{T_\Omega}\left(|P_{v^2}f^2(z_2^2)|^{(1+\epsilon)}\Delta^{v^2}(\Im z^2)\right)^{3\epsilon} |P_{v^2}f^2(z^2)|^{(1+\epsilon)}$$

$$\Delta^{v^2-\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)dV(z^2)$$

$$\leq (1+\epsilon)\|f^2\|^{(3\epsilon+3\epsilon^2)}_{L^{(1+\epsilon)}_{v^2}}\int_{T_\Omega} |P_{v^2}f^2(z^2)|^{(1+\epsilon)}\Delta^{v^2-\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)dV(z^2)$$

$$(1+\epsilon)\int_{T_\Omega}\|f^2\|^{(1+4\epsilon+\epsilon^2)}_{L^{(1+\epsilon)}_{v^2}}$$

Proposition (1.2): Let $(v^2,(1+\epsilon)) \in \sigma\ for\ 0\leq\epsilon\leq 1+\epsilon$.Suppose $P_{v^2_{(1+3\epsilon)}}$ is bounded on $L^{(1+\epsilon)}_{{v^2}_{(1+3\epsilon)}}(T_\Omega)$ for all $\epsilon = 0, \dots, (1+\epsilon)$.Then fro any $l\ \in \mathbb{N}$ we have

$$\int_{T_\Omega}\prod_{\epsilon=0}^{(1+\epsilon)}\left[\left|\left|P_{v^2_{(1+3\epsilon)}}\right| f^2(z^2)\right|^{(1+4\epsilon+\epsilon^2)}(\Im z^2)\right]\frac{dV(z^2)}{\Delta^{2\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)} \leq (1+\epsilon)\prod_{\epsilon=0}^{(1+\epsilon)}\|f^2_{(1+3\epsilon)}\|^{l_{(1+\epsilon)}}_{L^{(1+\epsilon)}_{v^2}}$$

Proof: Using the above proposition. $Hölder's$ inequality and Lemma (2.5) we obtain

$$\int_{T_\Omega}\prod_{\epsilon=0}^{(1+\epsilon)}\left[\left|P_{v^2_{(1+3\epsilon)}}\right| f^2(z^2)\right|^{(1+4\epsilon+\epsilon^2)}\Delta^{l{v^2}_{(1+3\epsilon)}+l\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)\right]\frac{dV(z^2)}{\Delta^{2\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)} \leq$$

$$(1+\epsilon)\prod_{\epsilon=0}^{(1+\epsilon)}\|f^2_{(1+3\epsilon)}\|^{(l-1)(1+\epsilon)}\ L^{(1+\epsilon)}_{v^2_{(1+\epsilon)}}$$

$$\int_{T_\Omega}\prod_{\epsilon=0}^{(1+\epsilon)}\left[\left|P_{v^2_{(1+3\epsilon)}}\right| f^2(z^2)\right|^{(1+\epsilon)}\Delta^{v^2_{(1+\epsilon)}+\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)\right]\frac{dV(z^2)}{\Delta^{2\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)} \leq$$

$$(1+\epsilon)\prod_{\epsilon=0}^{(1+\epsilon)}\|(f^2)_{(1+\epsilon)}\|^{(l-1)(1+\epsilon)}_{L^{(1+\epsilon)}_{v^2_{(1+3\epsilon)}}}\prod_{\epsilon=0}^{(1+\epsilon)}\left(\int_{T_\Omega}|P_{v^2}f^2(z^2)|^{(1+\epsilon)^2}\Delta^{(1+\epsilon)v^2_{(1+3\epsilon)}+\left(\frac{(1+\epsilon)^2}{1+2\epsilon}\right)}(\Im z^2)dV(z^2)\right)^{1/(1+\epsilon)}$$

$$\leq (1+\epsilon)\prod_{\epsilon=0}^{(1+\epsilon)}\left\|f_{(1+3\epsilon)}^2\right\|_{L_{v_{(1+\epsilon)}^2}^{(1+\epsilon)}}^{l_{(1+\epsilon)}}$$

It is well-know that the operator ■ satisfies the following boundedness estimate

$$\|\blacksquare f^2\|_{A_{v^2+(1+\epsilon)}^{(1+\epsilon)}} \leq c\|f^2\|_{A_{v^2}^{(1+\epsilon)}}$$

It is well-follows, using $H\ddot{o}lder's$inequality, that for $\epsilon \geq 0$ and$\epsilon = 0$

$$\int_{T_\Omega}|\blacksquare f^2(z^2)|^{(1+2\epsilon)}|f^2(z^2)|^{-\epsilon}\Delta^{v^2+(1+2\epsilon)-\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)dV(z^2) \leq (1+\epsilon)\|f^2\|_{A_{v^2}^{(1+\epsilon)}}$$

We obtain a multifunctional version of the above estimate. We introduce the following operator, which we still denote by ■, defined for pointwise products of holomorphic square functions:

$$\blacksquare\left(f_1^2, \dots f_{(1+\epsilon)}^2\right) = \sum_{\epsilon=0}^{(1+\epsilon)} f_1^2 \dots f_\epsilon^2\left(\blacksquare f_{(1+\epsilon)}^2\right)f_{2+\epsilon}^2 \dots f_{(1+\epsilon)}^2.$$

We note that the■ inside the sum is the usual ■ as defined at the beginning. The next theorem generalizes (22), this idea appearted in [10](see [12]).

Theorem (1.9): Let$v^2 > \left(\frac{2-\epsilon}{1+2\epsilon}\right), \epsilon \geq 0$. Then there exist $\epsilon > 0$such that

$$\int_{T_\Omega}\left|\blacksquare\left(f_1^2, \dots f_{(1+\epsilon)}^2\right)\right|^{(1+2\epsilon)}\prod_{\epsilon=0}^{(1+\epsilon)}\left|f_{(1+\epsilon)}^2(z^2)\right|^{-\epsilon}\Delta^{(1+\epsilon)\left(v^2+\frac{1+\epsilon}{1+2\epsilon}\right)+(1+2\epsilon)}(\Im z^2)\frac{dV(z^2)}{\Delta^{2\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)}$$
$$\leq (1+\epsilon)\prod_{\epsilon=0}^{(1+\epsilon)}\left\|f_{(1+\epsilon)}^2\right\|_{A_{v^2}^{(1+\epsilon)}}^{(1+\epsilon)}$$

Proof: Using $Minkowski's$ inequality, the pointwise estimate for function in $A_{v^2}^{(1+\epsilon)}(T_\Omega)$ and the estimate (22) we obtain

$$\int_{T_\Omega}\left|\blacksquare\left(f_1^2, \dots f_{(1+\epsilon)}^2\right)\right|^{(1+2\epsilon)}\prod_{\epsilon=0}^{(1+\epsilon)}\left|f_{(1+\epsilon)}^2(z^2)\right|^{-\epsilon}\Delta^{(1+\epsilon)\left(v^2+\frac{1+\epsilon}{1+2\epsilon}\right)+(1+2\epsilon)}(\Im z^2)\frac{dV(z^2)}{\Delta^{2\left(\frac{1+\epsilon}{1+2\epsilon}\right)}(\Im z^2)} \leq c$$

$$\left(\sum_{\epsilon=0}^{(1+\epsilon)}\left(\int_{T_\Omega}\prod_{3\epsilon\neq\epsilon}^{(1+\epsilon)}\frac{\left|f_{(1+3\epsilon)}^2(z^2)\right|^{(1+2\epsilon)}\left|\blacksquare f_{(1+3\epsilon)}^2(z^2)\right|^{(1+2\epsilon)}}{\prod_{\epsilon=0}^{(1+\epsilon)}\left|f_{(1+3\epsilon)}^2(z^2)\right|^{-\varepsilon}\Delta^{(1+\epsilon)\left(v^2+\frac{1+\epsilon}{1+2\epsilon}\right)+(1+2\epsilon)}}\right)^{1/(1+2\epsilon)}\right)^{(1+2\epsilon)}$$

4. Paley-Wiener Representation and Embeddings:

We make use of Paley-Wiener theory to prove Theorem (1.3) and Theorem (1.4). We fix a measure $\mu = \mu_{(1+2\epsilon)}$where $(1+2\epsilon) \in \Xi$. We recall that $\mathcal{H}_\mu^2(T_\Omega)$ is a Hilbert space see [8]. Then

$$L_{(1+2\epsilon)^\star}^2(\Omega) = L^2\left(\Omega, \Delta_{(1+2\epsilon)^\star}^2(2\xi^2)d\xi^2\right) = L^2\left(\Omega, \Delta_{(1+2\epsilon)^\star}^2(2\xi^2)^{-1}d\xi^2\right)^{-1}.$$

The followingPaley-Wiener characterization of functions in $\mathcal{H}_\mu^2(T_\Omega)$ has been obtained in [8]

Theorem (1.10): For every$F^2 \in \mathcal{H}_\mu^2(T_\Omega)$there is an $F^2 \in L_{(1+2\epsilon)^\star}^2(T_\Omega)$ such that

Conversely, if $f^2 \in L^2_{(1+2\epsilon)^*}(\Omega)$ then the above integral converges absolutely to a function $F^2 \in \mathcal{H}^2_\mu(T_\Omega)$. Moreover $\|F^2\|_{\mathcal{H}^2_\mu} = \|f^2\|_{L^2_{(1+2\epsilon)^*}}$.

We only need to show the following result in proving Theorem (1.3) (see [5]).

Theorem (1.11): Let $(1+2\epsilon) \in \Xi, \mu = \mu_{(1+2\epsilon)}$ *for all* $0 \le \epsilon < \infty$ such that $\frac{(1+2\epsilon)^2}{2} > g_0^2$ we have

$$\mathcal{H}^2_\mu(T_\Omega) \hookrightarrow A_{\frac{(1+2\epsilon)^2}{2}}^{(1+2\epsilon)}(T_\Omega).$$

Proof: Let $F^2 \in \mathcal{H}^2_\mu(T_\Omega)$ by Theorem (1.10) there is an f^2 in $L^2_{(1+2\epsilon)^*}(T_\Omega)$ such that

$$F^2(z^2) = (1+\epsilon)_{(3+\epsilon)} \int_\Omega e^{i(x^2/\xi^2)} f^2(\xi^2) \Delta^*_{(1+2\epsilon)^*}(2\xi^2) d\,\xi^2, \qquad z^2 \in T_\Omega$$

It follow from Plancherel's theorem that

$$\int_{\mathbb{R}^{(3+\epsilon)}} |F^2(x^2+iy^2)|^2 dx^2 = (1+\epsilon) \int_\Omega e^{-2(y^2/\xi^2)} f^2(\xi^2) \Delta^*_{(1+2\epsilon)^*}(2\xi^2) d\,\xi^2, \qquad z^2 \in T_\Omega$$

Integration the $(1+2\epsilon)/2 - power$ of the left hand side of the above equality with respect to the measure $\Delta_{\left(\frac{(1+2\epsilon)^2}{2}\right)}(y^2)\Delta^{-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(y^2)dy^2$

and using $Minkowski's$ inequality for integrals and Lemma (2.1) we obtain

$$I = (1+\epsilon) \int_\Omega \left(\int_\Omega e^{i(x^2(1+2\epsilon)/\xi^2)} |f^2(\xi^2)|^2 \Delta^*_{(1+2\epsilon)^*}(\xi^2) d\,\xi^2 \right)^2 \Delta^*_{(1+2\epsilon)^*}(y^2) \frac{dy^2}{\Delta^{-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(y^2)}$$

$$\le (1+\epsilon) \left(\int_\Omega \left(\int_\Omega e^{-(1+2\epsilon/\xi^2)} \Delta_{(1+2\epsilon)^*}(y^2) \frac{dy^2}{\Delta^{-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(y^2)} \right)^{2/(1+2\epsilon)} |f^2\xi^2|^2 \Delta^*_{(2+4\epsilon)^*}(\xi^2) d\xi^2 \right)^{(1+2\epsilon)/2}$$

$$= C \left(\int_\Omega \Delta^*_{-(1+2\epsilon)^*}(\xi^2) |f^2(\xi^2)|^2 \Delta^*_{(2+4\epsilon)^*}(\xi^2) d\xi^2 \right)^{(1+2\epsilon)/2} = C \|f^2\|^{(1+2\epsilon)}_{L^{(1+2\epsilon)}_{(1+2\epsilon)^*}}$$

where

$$I = \int_\Omega \left(\int_{\mathbb{R}^{(3+\epsilon)}} |F^2(x^2+iy^2)|^2 \, dx^2 \right)^{2/(1+2\epsilon)} \Delta_{(1+2\epsilon)^*}(y^2) \frac{dy^2}{\Delta^{-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}(y^2)} = \|f^2\|^{(1+2\epsilon)}_{A^{(2+4\epsilon)}_{(1+2\epsilon)^*}}$$

Finally we need the following Paley-Wiener construction of functions in the Bergman space $A^{2,(1+2\epsilon)}_{v^2}$(see [12]).

Lemma (2.8): Let $\epsilon \ge 0$ and $v^2 \in \mathbb{R}^{(1+2\epsilon)}$. If f^2 is in the space

$L^2_{2\left(1-\frac{1}{(1+2\epsilon)}\right)v^{2*}}(\Omega) = L^2\left(\Omega, \Delta^2_{2\left(\frac{1+4\epsilon}{1+2\epsilon}\right)v^{2*}}(2\xi^2)d\xi^2\right)$, then the function F^2 defined by

$$F^2(z^2) = \frac{1}{(2\pi)^{\frac{3+\epsilon}{2}}} \int_\Omega e^{-(x^2/\xi^2)} f^2(\xi^2) \Delta^*_{(1+2\epsilon)^*}(2\xi^2) d\,\xi^2, z^2 \in T_\Omega$$

belongs to $A^{2,(1+2\epsilon)}_{(1+2\epsilon)^*}(T_\Omega)$.

Proof: The estimation of the $L^{2,(1+2\epsilon)}_{v^2}$-norm of the integral in (24)proceeds exactly as in the previous theorem and one obtains

$$\|F^2\|_{A^{2,(1+2\epsilon)}_{v^2}} \leq (1+\epsilon)\|f^2\|_{L^2_{\left(\frac{1+4\epsilon}{1+2\epsilon}\right)v^{2*}}}$$

Thus, we only have to prove that for any $f^2 \in L^2_{2\left(\frac{2\epsilon}{1+2\epsilon}\right)v^{2*}}(\Omega)$ the integral in (24) converges absolutely to a holomorphic function $F^2(z^2)$ on T_Ω. It suffices to prove this at the point $z^2 = ie$. Using $\ddot{H}older's$ inequality and Lemma (2.1) we obtain

$$\int_\Omega e^{-(e/\xi^2)}|f^2(\xi^2)|\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2 \leq \|f^2\|_{L^2_{2\left(\frac{2\epsilon}{1+2\epsilon}\right)v^{2*}}}\left(\int_\Omega e^{-2(e/\xi^2)}\,\Delta^2_{\left(\frac{2}{1+\epsilon}\right)v^{2*}}(2\xi^2)d\xi^2\right)^{1/2}$$

$$= \|f^2\|_{L^2_{\left(\frac{4\epsilon-1}{1+2\epsilon}\right)v^{2*}}} 2^{-\left(\frac{3+\epsilon}{1+2\epsilon}\right)}\Gamma_\Omega\left(\frac{2}{(1+2\epsilon)}v^{2\star} + \frac{3+\epsilon}{1+2\epsilon}\right)^{1/2}$$

and this clearly finite.

We now give a proof of the following result, which implies Theorem (1.4) (see [12]).

Theorem (1.12): Let $(1+2\epsilon) \in \Xi, \mu = \mu_{(1+2\epsilon)}$for all $0 \leq \epsilon < \infty$we have

$$\mathcal{H}^2_\mu(T_\Omega) \hookrightarrow A^{(1+2\epsilon)}_{\frac{(1+2\epsilon)}{4}\left(\frac{5+9\epsilon+4\epsilon^2}{1+2\epsilon}\right)}(T_\Omega).$$

Proof: Given F^2 in$\mathcal{H}^2_\mu(T_\Omega)$ we need to show that F^2belongs to $A^{2,(1+2\epsilon)/2}_{\frac{(1+2\epsilon)}{4}\left(\frac{5+9\epsilon+4\epsilon^2}{1+2\epsilon}\right)}(T_\Omega)$ By Theorem (1.10) there exists $L^2_{(1+2\epsilon)^*}(T_\Omega)$such that

$$F^2(z^2) = (1+\epsilon)\int_\Omega e^{i(x^2/\xi^2)}f^2(\xi^2)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2, z^2 \in T_\Omega$$

Using this Paley-Wiener representation we get

$$F^2(z^2) = (1+\epsilon)^2_{(3+\epsilon)}\int_{\Omega\times\Omega} e^{i(x^2/2\xi^2)}f^2(\xi^2)f^2(1+2\epsilon)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2 d(1+2\epsilon),$$

$$= (1+\epsilon)^2_{(3+\epsilon)}\int_\Omega\int_{\Omega\times\Omega} e^{i(x^2/u^2)}f^2(u^2-\xi^2)f^2(\xi^2)\Delta^*_{(1+2\epsilon)^*}\left(2\left((u^2-\xi^2\right)\right)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2 d(1+2\epsilon),$$

$$= (1+\epsilon)^2_{(3+\epsilon)}\int_\Omega e^{i(x^2/u^2)}g^2(u^2)du^2,$$

where

$$g^2(u^2) = \int_{\Omega\cap(u^2-\Omega)} f^2(u^2-\xi^2)\Delta^*_{(1+2\epsilon)^*}\left(2\left((u^2-\xi^2\right)\right)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2$$

By Lemma (2.8) in suffices to prove that $g^2(u^2)\Delta^*_{-\frac{1+2\epsilon}{4}\left(\frac{3+8\epsilon+8\epsilon^2}{1+2\epsilon}\right)}(u^2)\ is\ L^2_{\left(\frac{2\epsilon-1\epsilon}{2}\right)\left(\frac{3+8\epsilon+8\epsilon^2}{1+2\epsilon}\right)}(\Omega)$

or, equivalently, that is in $L^2_{-\left(\frac{5+9\epsilon+4\xi^2}{1+2\epsilon}\right)}(\Omega)$. We start with a pointwise estimate of$g^2(u^2)$. $H\ddot{o}lder's$ inequality and Lemma (2.2) we obtain

$$|g^2(u^2)|^2 \leq \left(\int_{\Omega\cap(u^2-\Omega)}|f^2(u^2-\xi^2)||f^2(\xi^2)|\Delta^*_{(1+2\epsilon)^*}\left(2\left((u^2-\xi^2\right)\right)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2\right)^2$$

$$\leq \left(\int_{\Omega\cap(\mathrm{u}^2-\Omega)} |f^2(u^2-\xi^2)|^2|f^2(\xi^2)|^2\Delta^*_{(1+2\epsilon)^*}\left(2\left((u^2-\xi^2\right)\right)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\,\xi^2\right)^2$$

$$\times\left(\int_{\Omega\cap(\mathrm{u}^2-\Omega)} \Delta^*_{(1+2\epsilon)^*}\left(2\left((u^2-\xi^2\right)\right)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2\right)$$

$$= (1+\epsilon)\Delta^*_{\left(2s^*+\frac{3+\epsilon}{1+2\epsilon}\right)}(u^2)\left(\int_{\Omega\cap(u^2-\Omega\,)} |f^2(u^2\;-\xi^2)||f^2(\xi^2)|\Delta^*_{(1+2\epsilon)^*}\left(2\left((u^2-\xi^2\right)\right)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2\right)$$

It easily follows that

$$\int_\Omega \frac{|g^2(u^2)|^2\mathrm{du}^2}{\Delta^*_{\left(2(1+2\epsilon)^*+\frac{3+\epsilon}{1+2\epsilon}\right)}}$$

$$\leq (1+\epsilon)\int_\Omega\int_{\Omega\cap(u^2-\Omega\,)} |f^2(u^2-\xi^2)||f^2(\xi^2)|\Delta^*_{(1+2\epsilon)^*}\left(2\left((u^2-\xi^2\right)\right)\Delta^*_{(1+2\epsilon)^*}(2\xi^2)d\xi^2\mathrm{d}u^2$$

$$= (1+\epsilon)\|f^2\|^4_{L^2_{(1+2\epsilon)^*}} = (1+\epsilon)\|F^2\|^4_{\mathcal{H}^2_\mu}$$

and the proof is complete.

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