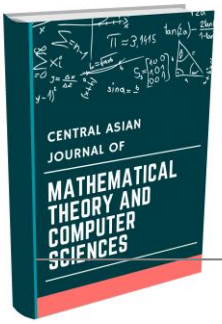

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K-STEP BLOCK HYBRID NYSTRÖM-TYPE METHOD FOR THE SOLUTION OF BRATU PROBLEM WITH IMPEDANCE BOUNDARY CONDITION

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Abstract

Using a carefully selected step and off-step points produced from the Bhaskara Cosine approximation formula and Gauss-Lobato grid points with constant step size, this numerical study presents the Block Hybrid Nyström-type method of order seven for solving the second order Bratu problem with impedance boundary conditions at two-point concurrently. Without the use of a combination of predictor-corrector mode and or the shooting approach, the solution is achieved immediately without having to convert it to a system of first order ordinary differential equations. The diffusion of heat produced by the application that imposed the impedance conditions is the focus of a numerically tested problem. The present findings showed that the proposed method produce an efficient performance in terms of accuracy, and error when compared with the existing methods. The convergence and stability properties of the BHNTM are discussed.

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1.0. Introduction

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either analytically or numerically. There has recently much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytic or numerical, to nonlinear models, Wang (1988). This paper, focuses on the numerical method for solving the second order two-point boundary value problem of Bratu-type connected with impedance conditions. In general, the following boundary value problem is given as:

$$y''(x) = f(x, y, y') \quad \text{for} \quad a \le x \le b \tag{1}$$

with

$$\begin{aligned} q_1 y'(a) + q_2 y(a) &= \xi \\ q_3 y'(b) + q_4 y(b) &= \psi \end{aligned} \tag{2}$$

where $a, b, q_1, q_2, q_3, q_4, \xi,$ and ψ are constants and $q_1, q_2, q_3,$ and q_4 are nonzero. Most recently, attention has been given to solve equation (1) by conventional methods of Adomian decomposition and its modifications, Runge-Kutta methods, multistep methods and some block methods developed (see, Sunday *et al*, (2022), Mishra *et al*, (2021), Ogundiran *et al*, (2019), and Singh *et al*, (2019), Sunday *et al*, (2014)). Using these conventional techniques, one prominent approach in a transformation of these problems into a system of reduced order differential equations. The approach is good but lack a stable numerical scheme for approximating these problems. Efficient numerical schemes are needed due to the fall in the stability of these problems due to their transformations. Numerous existing researches of hybrid block method used the approach of predictor-corrector or guess in selecting their hybrid points. Toraman *et al.* (2019), Noor and Noor (2021). The motivation of this study is to produce technique whose hybrid points are obtained rather than estimated. Without loss of generality, the introduction of a unique way of obtaining the hybrid points in the selected intervals with the aim of improving the solution of the existing methods. This research aims to solve the Bratu-type problem with impedance boundary conditions which is of the form (1) on an interval $[x_0, x_n]$ using the continuous implicit k-step Block Hybrid Nyström-type method.

Functional value and derivatives of the solutions are provided for Impedance type in (2). Boundary value issues will be susceptible to the Neumann condition if only derivative values are present; otherwise, condition (2) is known as the Dirichlet type. According to Akano & Fakinlede (2015), Impedance boundary conditions are known as Robin boundary conditions and the convective boundary conditions respectively, and they occur in a variety of application fields such as electromagnetic problems and heat transfer problems. The Bernoulli polynomial together with Galerkin approximation in solving linear and nonlinear Robin boundary condition problems had been studied by Islam & Shirin (2011).

Renowned researchers developed Adomian decomposition techniques for solving analytical stage and numerical simulations, including Duan *et al.* (2013) and Rach *et al.* (2016). While this was going on, the boundary value problems in question were discretized by Bhatta & Sastri (1995) and Lang & Xu (2012) and then solved using symmetric global (continuous) splines and the Quintic B-Spline collocation approach, respectively.

As mentioned by Fatunla (1995), in order to reduce computing costs, the proposed algorithm's development must at the very least provide this capability of simultaneously generating solutions at several sites. These authors' works like Nasir *et al.* (2018), Phang *et al.* (2012), Majid *et al.* (2013), and Omar & Adeyeye (2016), all provide evidence for this implementation. At two moments simultaneously, they all got close to the answer to (1). This study was motivated by the benefits of the results from their conversation.

Researchers such as Majid (2004) and Zawawi *et al* (2012) adopted the Diagonal block method for solving first order differential equations. Solving second order ordinary differential equations using diagonal block method has been discussed in Zinuddin *et al.* (2014). In this research, the extension of the derivation of Jator & Manathunga (2018) to obtain the formulation of direct integration for solving second order differential equations with specially selected step and off-step points. The new formulae obtained have been implemented to solve the boundary value problems.

The organization of this paper is as follows: Section 2 presents the derivation of the two-point Hybrid Block Nyström-Type method. Section 3 explains the analysis of the method including the order, consistency, and stability. Section 4 presents the validation and clear overview with tested problem. Section 5 concludes the findings from the study.

2. Mathematical Formulation

In order to obtain the numerical formula for the approximate solution of (1), the function

$$y(x) \approx Y(x) = \sum_{j}^{c+i-1} a_j x^j \tag{3}$$

is considered as the basis where x is continuous within the interval [a, b], c and i represents, collocation and interpolation points respectively. Variables a_j are coefficients to be determined distinctly. The second derivatives of (3) equated to (3) is given as

$$y''(x) \approx Y''(x) = \sum_{j}^{c+i-1} j\left(j-1\right) a_j x^{j-2} = f\left(x, y, y'\right) \tag{4}$$

Evaluating (3) at $x = x_{n+v}$, $v = \frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74}$, using $t = \frac{x - x_{n+k}}{h}$ yields the following interpolation and collocation matrix

$$XA = B \tag{5}$$

where $A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ . \\ a_8 \end{bmatrix}$, $B = \begin{bmatrix} y_n \\ y'_n \\ f_n \\ f_{n+\frac{5}{74}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{69}{74}} \\ f_{n+1} \end{bmatrix}$,

$$X = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\
0 & 1 & 2x_n & 3x_n{}^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 \\
0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\
0 & 0 & 2 & 6x_{n+\frac{5}{74}} & 12x^2_{n+\frac{5}{74}} & 20x^3_{n+\frac{5}{74}} & 30x^4_{n+\frac{5}{74}} & 42x^5_{n+\frac{5}{74}} & 56x^6_{n+\frac{5}{74}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{4}} & 12x^2_{n+\frac{1}{4}} & 20x^3_{n+\frac{1}{4}} & 30x^4_{n+\frac{1}{4}} & 42x^5_{n+\frac{1}{4}} & 56x^6_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} & 42x^5_{n+\frac{1}{2}} & 56x^6_{n+\frac{1}{2}} \\
0 & 0 & 2 & 6x_{n+\frac{3}{4}} & 12x^2_{n+\frac{3}{4}} & 20x^3_{n+\frac{3}{4}} & 30x^4_{n+\frac{3}{4}} & 42x^5_{n+\frac{3}{4}} & 56x^6_{n+\frac{3}{4}} \\
0 & 0 & 2 & 6x_{n+\frac{69}{74}} & 12x^2_{n+\frac{69}{74}} & 20x^3_{n+\frac{69}{74}} & 30x^4_{n+\frac{69}{74}} & 42x^5_{n+\frac{69}{74}} & 56x^6_{n+\frac{69}{74}} \\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 & 56x_{n+1}^6
\end{bmatrix}$$

Using Maple 2015 Software, the above system of nonlinear equations is solved via the matrix inversion algorithm. The unknown coefficients a_j's, are then substituted into (3) to obtain the continuous form of the proposed Hybrid Block Nyström -type Method **(HBNTM5)**:

$$y(x)=\alpha_0(x)y_n+\alpha_1(x)hy'_n+h^2\begin{pmatrix}\beta_n f_n+\beta_{n+\frac{5}{74}}f_{n+\frac{5}{74}}+\beta_{n+\frac{1}{4}}f_{n+\frac{1}{4}}+\beta_{n+\frac{1}{2}}f_{n+\frac{1}{2}}\\+\beta_{n+\frac{3}{4}}f_{n+\frac{3}{4}}+\beta_{n+\frac{69}{74}}f_{n+\frac{69}{74}}+\beta_{n+1}f_{n+1}\end{pmatrix}, \qquad (6)$$

and

$$y'(x)=y'_n+h\left(\beta_n f_n+\beta_{n+\frac{5}{74}}f_{n+\frac{5}{74}}+\beta_{n+\frac{1}{4}}f_{n+\frac{1}{4}}+\beta_{n+\frac{1}{2}}f_{n+\frac{1}{2}}+\beta_{n+\frac{3}{4}}f_{n+\frac{3}{4}}+\beta_{n+\frac{69}{74}}f_{n+\frac{69}{74}}+\beta_{n+1}f_{n+1}\right) \qquad (7)$$

where $\alpha_0(x)$, $\alpha_1(x)$ and $\beta_n(x)$ are continuous coefficients that are distinctly determined as presented in Table 1:

Table 1: Coefficients of y_n, y'_n, and f_{n+j} in (8) to (19) for $j=\frac{5}{74},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{69}{74},1.$

y_n	y'_n	f_n	$f_{n+\frac{5}{74}}$	$f_{n+\frac{1}{4}}$	$f_{n+\frac{1}{2}}$	$f_{n+\frac{3}{4}}$	$f_{n+\frac{69}{74}}$	f_{n+1}
1	$\frac{5}{74}$	$\frac{58052434225}{44612850799692}$	$\frac{15063275}{13592395776}$	$-\frac{2933478125}{16325717140467}$	$\frac{24930298125}{294258030395392}$	$-\frac{24485719375}{440794362792609}$	$\frac{7823810875}{177258433314816}$	$-\frac{6595375}{323281527534}$
1	$\frac{1}{4}$	$\frac{348899}{89026560}$	$\frac{68719861387}{3069160980480}$	$\frac{449}{77568}$	$-\frac{92961}{73400320}$	$\frac{155707}{219905280}$	$-\frac{8806682539}{16573469294592}$	$\frac{7177}{29675520}$
1	$\frac{1}{2}$	$\frac{211}{28980}$	$\frac{7142427571}{129480228864}$	$\frac{48253}{859005}$	$\frac{115}{16384}$	$-\frac{241}{286335}$	$\frac{69343957}{215800381440}$	$-\frac{1}{8694}$
1	$\frac{3}{4}$	$\frac{37377}{3297280}$	$\frac{29332493811}{341017886720}$	$\frac{103833}{904960}$	$\frac{4690143}{73400320}$	$\frac{449}{77568}$	$-\frac{762783527}{1023053660160}$	$\frac{729}{3297280}$
1	$\frac{69}{74}$	$\frac{4946912289}{359201697260}$	$\frac{17400136203}{158577950720}$	$\frac{1411081172199}{9069842855815}$	$\frac{165964330541889}{1471290151976960}$	$\frac{1127174082469}{27209528567445}$	$\frac{15063275}{13592395776}$	$\frac{10840797}{35920169726}$
1	1	$\frac{643}{43470}$	$\frac{69343957}{586414080}$	$\frac{48976}{286335}$	$\frac{4671}{35840}$	$\frac{48976}{859005}$	$\frac{69343957}{8092514304}$	0
0	1	$\frac{132171928615}{4823010897264}$	$\frac{13113969205}{299423029248}$	$-\frac{61021705750}{11913361156557}$	$\frac{1181523375}{497057483776}$	$-\frac{18479790250}{11913361156557}$	$\frac{368391325}{299423029248}$	$-\frac{2741432975}{4823010897264}$
0	1	$\frac{23579}{5564160}$	$\frac{199779940117}{1294802288640}$	$\frac{1422961}{13744080}$	$-\frac{20709}{1146880}$	$\frac{139651}{13744080}$	$-\frac{9916185851}{1294802288640}$	$\frac{19421}{5564160}$
0	1	$\frac{277}{12880}$	$\frac{901471441}{8092514304}$	$\frac{217162}{859005}$	$\frac{4671}{35840}$	$-\frac{2362}{95445}$	$\frac{69343957}{4495841280}$	$-\frac{467}{69552}$
0	1	$\frac{2329}{206080}$	$\frac{2149662667}{15985213440}$	$\frac{36973}{169680}$	$\frac{319653}{1146880}$	$\frac{63389}{509040}$	$-\frac{1317535183}{47955640320}$	$\frac{435}{41216}$
0	1	$\frac{596477423}{38832615920}$	$\frac{100924207}{803604480}$	$\frac{168853796338}{735392663985}$	$\frac{641903629419}{2485287418880}$	$\frac{514439521514}{2206177991955}$	$\frac{200151221}{2410813440}$	$-\frac{489781527}{38832615920}$
0	1	$\frac{643}{43470}$	$\frac{2565726409}{20231285760}$	$\frac{195904}{859005}$	$\frac{4671}{17920}$	$\frac{195904}{859005}$	$\frac{2565726409}{20231285760}$	$\frac{643}{43470}$

assuming that y_n is the numerical approximation to the analytical solution $y(x_n)$, y'_n is the numerical approximation to $y'(x_n)$ and f_{n+j} is the numerical approximation to $f\left(x_{n+hj},y_{n+hj},y'_{n+hj}\right)$. The main methods are obtained by evaluating (6) to give the following:

$$y_{n+\frac{5}{74}}=y_n+\frac{5}{74}hy'_n+h^2\begin{pmatrix}\frac{58052434225}{44612850799692}f_n+\frac{15063275}{13592395776}f_{n+\frac{5}{74}}-\frac{2933478125}{16325717140467}f_{n+\frac{1}{4}}+\frac{24930298125}{294258030395392}f_{n+\frac{1}{2}}\\-\frac{24485719375}{440794362792609}f_{n+\frac{3}{4}}+\frac{7823810875}{177258433314816}f_{n+\frac{69}{74}}-\frac{6595375}{323281527534}f_{n+1}\end{pmatrix} \qquad (8)$$

$$y_{n+\frac{1}{4}} = y_n + \frac{1}{4}hy'_n + h^2\left(\begin{array}{l}\frac{348899}{89026560}f_n + \frac{68719861387}{3069160980480}f_{n+\frac{5}{74}} + \frac{449}{77568}f_{n+\frac{1}{4}} - \frac{92961}{73400320}f_{n+\frac{1}{2}} + \frac{155707}{219905280}f_{n+\frac{3}{4}} \\ -\frac{8806682539}{16573469294592}f_{n+\frac{69}{74}} + \frac{7177}{29675520}f_{n+1}\end{array}\right) \quad (9)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + h^2\left(\begin{array}{l}\frac{211}{28980}f_n + \frac{7142427571}{129480228864}f_{n+\frac{5}{74}} + \frac{48253}{859005}f_{n+\frac{1}{4}} + \frac{115}{16384}f_{n+\frac{1}{2}} - \frac{241}{286335}f_{n+\frac{3}{4}} + \frac{69343957}{215800381440}f_{n+\frac{69}{74}} \\ -\frac{1}{8694}f_{n+1}\end{array}\right) \quad (10)$$

$$y_{n+\frac{3}{4}} = y_n + \frac{3}{4}hy'_n + h^2\left(\begin{array}{l}\frac{37377}{3297280}f_n + \frac{29332493811}{341017886720}f_{n+\frac{5}{74}} + \frac{103833}{904960}f_{n+\frac{1}{4}} + \frac{4690143}{73400320}f_{n+\frac{1}{2}} + \frac{449}{77568}f_{n+\frac{3}{4}} \\ -\frac{762783527}{1023053660160}f_{n+\frac{69}{74}} + \frac{729}{3297280}f_{n+1}\end{array}\right) \quad (11)$$

$$y_{n+\frac{69}{74}} = y_n + \frac{69}{74}hy'_n + h^2\left(\begin{array}{l}\frac{4946912289}{359201697260}f_n + \frac{17400136203}{158577950720}f_{n+\frac{5}{74}} + \frac{1411081172199}{9069842855815}f_{n+\frac{1}{4}} + \frac{165964330541889}{147129015976960}f_{n+\frac{1}{2}} \\ +\frac{1127174082469}{27209528567445}f_{n+\frac{3}{4}} + \frac{15063275}{13592395776}f_{n+\frac{69}{74}} + \frac{10840797}{35920169726}f_{n+1}\end{array}\right) \quad (12)$$

$$y_{n+1} = y_n + hy'_n + h^2\left(\frac{643}{43470}f_n + \frac{69343957}{586414080}f_{n+\frac{5}{74}} + \frac{48976}{286335}f_{n+\frac{1}{4}} + \frac{4671}{35840}f_{n+\frac{1}{2}} + \frac{48976}{859005}f_{n+\frac{3}{4}} + \frac{69343957}{8092514304}f_{n+\frac{69}{74}}\right) \quad (13)$$

The additional methods are obtained by evaluating (7) to give the following:

$$y'_{n+\frac{5}{74}} = y'_n + h\left(\begin{array}{l}\frac{132171928615}{4823010897264}f_n + \frac{13113969205}{299423029248}f_{n+\frac{5}{74}} - \frac{61021705750}{11913361156557}f_{n+\frac{1}{4}} + \frac{1181523375}{497057483776}f_{n+\frac{1}{2}} - \frac{18479790250}{11913361156557}f_{n+\frac{3}{4}} \\ +\frac{368391325}{299423029248}f_{n+\frac{69}{74}} - \frac{2741432975}{4823010897264}f_{n+1}\end{array}\right) \quad (14)$$

$$y'_{n+\frac{1}{4}} = y'_n + h\left(\begin{array}{l}\frac{23579}{5564160}f_n + \frac{199779940117}{1294802288640}f_{n+\frac{5}{74}} + \frac{1422961}{13744080}f_{n+\frac{1}{4}} - \frac{20709}{1146880}f_{n+\frac{1}{2}} + \frac{139651}{13744080}f_{n+\frac{3}{4}} - \frac{9916185851}{1294802288640}f_{n+\frac{69}{74}} \\ +\frac{19421}{5564160}f_{n+1}\end{array}\right) \quad (15)$$

$$y'_{n+\frac{1}{2}} = y'_n + h\left(\frac{277}{12880}f_n + \frac{901471441}{8092514304}f_{n+\frac{5}{74}} + \frac{217162}{859005}f_{n+\frac{1}{4}} + \frac{4671}{35840}f_{n+\frac{1}{2}} - \frac{2362}{95445}f_{n+\frac{3}{4}} + \frac{69343957}{4495841280}f_{n+\frac{69}{74}} - \frac{467}{69552}f_{n+1}\right) \quad (16)$$

$$y'_{n+\frac{3}{4}} = y'_n + h\left(\frac{2329}{206080}f_n + \frac{2149662667}{15985213440}f_{n+\frac{5}{74}} + \frac{36973}{169680}f_{n+\frac{1}{4}} + \frac{319653}{1146880}f_{n+\frac{1}{2}} + \frac{63389}{509040}f_{n+\frac{3}{4}} - \frac{1317535183}{47955640320}f_{n+\frac{69}{74}} + \frac{435}{41216}f_{n+1}\right) \quad (17)$$

$$y'_{n+\frac{69}{74}} = y'_n + h\left(\begin{array}{l}\frac{596477423}{38832615920}f_n + \frac{10024207}{803604480}f_{n+\frac{5}{74}} + \frac{168853796338}{735392663985}f_{n+\frac{1}{4}} + \frac{641903629419}{2485287418880}f_{n+\frac{1}{2}} + \frac{514439521514}{2206177991955}f_{n+\frac{3}{4}} \\ +\frac{200151221}{2410813440}f_{n+\frac{69}{74}} - \frac{489781527}{38832615920}f_{n+1}\end{array}\right) \quad (18)$$

$$y'_{n+1} = y'_n + h\left(\frac{643}{43470}f_n + \frac{2565726409}{20231285760}f_{n+\frac{5}{74}} + \frac{195904}{859005}f_{n+\frac{1}{4}} + \frac{4671}{17920}f_{n+\frac{1}{2}} + \frac{195904}{859005}f_{n+\frac{3}{4}} + \frac{2565726409}{20231285760}f_{n+\frac{69}{74}} + \frac{643}{43470}f_{n+1}\right) \quad (19)$$

3.0. Analysis of the Proposed Hybrid Block Nystrom-Type Method

3.1. Local truncation error

The local truncation error associated with a second order differential equation by the difference operator:

$$\eta\left[y(x);h\right]=\sum_{i=0}^{k}\left[a_i y\left(x_n+ih\right)-h^2\beta_i f\left(x_n+ih\right)\right] \tag{20}$$

where $y(x)$ is an arbitrary function, continuously differentiable on the interval $\left[x_n,x_{n+1}\right]$. Expanding the expression in (20) in Taylor series about the point x, gives:

$$\eta\left[y(x);h\right]=C_0y(x)+C_1hy'(x)+C_2h^2y''(x)+\ldots+C_{p+1}h^{p+2}y^{p+2}(x)+\ldots+C_qh^qy^q(x) \tag{21}$$

where vectors

$$\left.\begin{aligned}
C_0&=\sum_{i=0}^{k}\alpha_i,\\
C_1&=\sum_{i=0}^{k}i\alpha_i,\\
C_2&=\frac{1}{2!}\sum_{i=0}^{k}i\alpha_i-\beta_i,\\
C_p&=\frac{1}{p!}\sum_{i=0}^{k}i^p\alpha_i-p\left(p-1\right)\left(p-2\right)i^{p-2}\beta_i
\end{aligned}\right\} \tag{22}$$

According to Lambert (1991), the method's order is $p+2$ if in (21)

$C_0=C_1=C_2=\ldots=C_P=C_{p+1}=0,$ and $C_{p+2}\neq 0.$ Therefore, C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{p+2}\left(x_n\right)$ is the principal local truncation error at the point x_n .

The local truncation error over the selected interval is defined to be the difference between the result Y_{n+1} of the proposed method supposing all the previous values are accurate, and the solution is given by $\hat{Y}_{n+1}$. Hence, the local truncation error

$\eta_{n+1}=\hat{Y}_{n+1}-Y_{n+1},$

where

$$\hat{Y}_{n+1}=\begin{bmatrix} y\left(x_n+\frac{5}{74}h\right),y\left(x_n+\frac{1}{4}h\right),y\left(x_n+\frac{1}{2}h\right),\ldots,y\left(x_n+1\right),\\ y'\left(x_n+\frac{5}{74}h\right),y'\left(x_n+\frac{1}{4}h\right),y'\left(x_n+\frac{1}{2}h\right),\ldots,y'\left(x_n+1\right)\end{bmatrix}^T. \tag{20}$$

Supposing that $y\left(x_n\right)$ is sufficiently differentiable, the expressions $y\left(x_n+v_ih\right),\ y'\left(x_n+v_ih\right)$ and $f\left(x_n+v_ih\right)=y''\left(x_n+v_ih\right)$ can be written in Taylor series polynomial about the point x_n to obtain the following forms:

$$\left.\begin{aligned}
y\left(x_n+v_ih\right)&=\sum_{k=0}^{\infty}\left(v_ih\right)^k\frac{y^{(k)}\left(x_n\right)}{k!},\\
y'\left(x_n+v_ih\right)&=\sum_{k=0}^{\infty}\left(v_ih\right)^k\frac{y^{(k+1)}\left(x_n\right)}{k!},\\
y''\left(x_n+v_ih\right)&=\sum_{k=0}^{\infty}\left(v_ih\right)^k\frac{y^{(k+2)}\left(x_n\right)}{k!}.
\end{aligned}\right\} \tag{21}$$

Assuming all that the inputs to the method are exactly correct, then the main and additional methods becomes

$$\left.\begin{aligned}
y_{x_n+v_i} &= y(x_n)+v_i h y'(x_n)+h^2\beta_0(i)y''(x_n)+h^2\sum_{j=1}^{r+s-1}\beta_j(i)\left[\sum_{k=0}^{\infty}\frac{(v_i h)^k}{k!}y^{(k+2)}(x_n)\right],\\
&= y(x_n)+v_i h y'(x_n)+h^2\beta_0(i)y''(x_n)+h^2\sum_{k=0}^{\infty}\left[\sum_{j=1}^{r+s-1}\beta_j(i)\right](v_i h)^k\,\frac{y^{k+2}(x_n)}{k!},\\
&= y(x_n)+v_i h y'(x_n)+h^2\left[\sum_{j=0}^{r+s-1}\beta_j(i)\right]y''(x_n)+\sum_{k=1}^{\infty}\frac{\left[\sum_{j=1}^{r+s-1}v_i^{\,k}\beta_j(i)\right]}{k!}h^{k+2}y^{k+2}(x_n),\\
y'_{x_n+v_i} &= y'(x_n)+h\beta_0'(i)y''(x_n)+h\sum_{j=1}^{r+s-1}\beta_j'(i)\left[\sum_{k=0}^{\infty}\frac{(v_i h)^k}{k!}y^{(k+2)}(x_n)\right],\\
&= y'(x_n)+h\beta_0'(i)y''(x_n)+h\sum_{k=0}^{\infty}\left[\sum_{j=1}^{r+s-1}v_i^{\,k}\beta_j'(i)\right](h)^k\,\frac{y^{k+2}(x_n)}{k!},\\
&= y'(x_n)+hy''(x_n)\left[\sum_{j=0}^{r+s-1}\beta_j'(i)\right]+\sum_{k=1}^{\infty}\frac{\left[\sum_{j=1}^{r+s-1}v_i^{\,k}\beta_j{}'(i)\right]}{k!}h^{k+1}y^{k+2}(x_n),
\end{aligned}\right\}\tag{22}$$

the LTEs and order p is shown below.

For HBNTM with five (5) off-grid points ($w=5$), gives

$$\eta\left[y\left(x_n;h\right)\right]=\begin{cases}
-\dfrac{28132172125}{1207309213103041069056}h^{(9)}y^{(9)}(x)+\mathrm{O}\left(h^{10}\right)\\
\dfrac{147697}{651143046758400}h^{(9)}y^{(9)}(x)+\mathrm{O}\left(h^{10}\right)\\
\dfrac{43}{317940940800}h^{(9)}y^{(9)}(x)+\mathrm{O}\left(h^{10}\right)\\
\dfrac{351}{8038803046400}h^{(9)}y^{(9)}(x)+\mathrm{O}\left(h^{10}\right)\\
\dfrac{109474638723}{372626300340444774400}h^{(9)}y^{(9)}(x)+\mathrm{O}\left(h^{10}\right)\\
\dfrac{43}{158970470400}h^{(9)}y^{(9)}(x)+\mathrm{O}\left(h^{10}\right)
\end{cases}\tag{23}$$

showing that the proposed method with $w=5$ has order with at least $p\geq 7$.

$\hat{C}_0=\hat{C}_1=\ldots=\hat{C}_8=0$ and

$$\hat{C}_9=\begin{pmatrix}-\dfrac{28132172125}{1207309213103041069056},\dfrac{147697}{651143046758400},\dfrac{43}{317940940800},\\ \dfrac{351}{8038803046400},\dfrac{109474638723}{372626300340444774400},\dfrac{43}{158970470400}\end{pmatrix}.$$

3.2. Definition 1 (Consistency of the method)

The proposed (BHNTM) method is said to be consistent if the order of the method is greater than or equal to one, that is if $p\geq 1$.

In addition to:

i. $\rho(1)=0$ and

ii. $\rho'(1)=\sigma(1)$ where $\rho(z)$ and $\sigma(z)$ are first and second characteristic polynomial respectively defined as $\rho(z)=\sum_{j=0}^{k}\alpha_j z^j$ and $\sigma(z)=\sum_{j=0}^{k}\beta_j z^j$ where z satisfies

iii. $\sum_{j=0}^{k}\alpha_j=0$ ii. $\rho(1)=\rho'(1)=0$ and $\rho''(1)=2!\sigma(1)$

Definition 2 (Stability of the method)

Zero-stability is a property concerned with the method when limiting the step size to zero. Thus as the step size tends to zero in the main method (8) – (13), the following system of equations:

$$\left.\begin{aligned} y_{n+\frac{5}{74}} &= y_n \\ y_{n+\frac{1}{4}} &= y_n \\ y_{n+\frac{1}{2}} &= y_n \\ y_{n+\frac{3}{4}} &= y_n \\ y_{n+\frac{69}{74}} &= y_n \\ y_{n+1} &= y_n \end{aligned}\right\} \quad (24)$$

which can be written in matrix form as

$$A^0Y_i - A^1Y_{i-1}=0, \quad (25)$$

where

$$A^0=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\quad A^i=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix},\quad Y_i=\begin{pmatrix}y_{n+\frac{5}{74}}\\y_{n+\frac{1}{4}}\\y_{n+\frac{1}{2}}\\y_{n+\frac{3}{4}}\\y_{n+\frac{69}{74}}\\y_{n+1}\end{pmatrix},\quad Y_{i-1}=\begin{pmatrix}y_n\\y_n\\y_n\\y_n\\y_n\\y_n\end{pmatrix}$$

Following Ajinuhi *et al.* (2023), a method is said to be zero stable if the root r_i of the first characteristic polynomial $\rho(r)=\det\left|A^0r-A^i\right|$ does not exceed one ($|r_i|\le 1$).

The first characteristic polynomial of the BHNTM is given by

$$r^5\left[r-1\right]=0. \quad (26)$$

The roots of (26) are $r=0,0,0,0,0,1$ in which none of them is greater than one. Therefore, the BHNTM is zero-stable.

Definition 3 (Convergence of the method)

Let $y(x)$ denote the exact solution of the given boundary value problem and let $\{y_j\}_{j=0}^{N}$ be the approximations obtained with the developed numerical strategy. The method is said to be convergent of order p if, for a sufficiently small h, there exists a constant K independent of h, such that:

$$\max_{0\le j\le N}\left|y(x_j)-y_j\right|\le Kh^p.$$

Note that $\max_{0\le j\le N}\left|y(x_j)-y_j\right|\le Kh^p \to 0$ as $h\to 0$

Theorem 1 (Convergence theorem). Let $y(x)$ be the exact solution (1) with boundary conditions in (2), and $\{y_j\}_{j=0}^{N}$ the discrete solution provided by the proposed BHNTM, the proposed method is convergent to order seven.

Proof. Following Mufutau and Ramos (2021), the matrix D of dimension $9N \times 9N$ given by:

$$D=\begin{pmatrix} D_{1,1} & D_{1,2} & .. & ... & D_{1,2N} \\ . & . & . & . & . \\ . & . & . & . & . \\ D_{2N,1} & D_{2N,2} & ... & . & D_{2N,2N} \end{pmatrix},$$

where the elements $D_{i,j}$ are

The necessary and sufficient conditions for the proposed k-step Hybrid Block Nyström -Type method to be convergent are that, it must be consistent and zero stable according to Dahlquist (Lambert, 1973). Hence, by definitions 1 and 2 the method is convergent.

4.0. Implementation of the method

Considering the application of the derived schemes here to the Bratu's problem with Robin (Impedance) boundary condition for the efficiency and accuracy of the method implemented as block method.

Example 1: Consider the nonlinear second order Bratu problem

$$y''(x)+\lambda e^{-2y(x)}=0, \quad 0\le x\le 1$$

with boundary conditions:

$$y'(0)-y(0)=1 \text{ and } y'(0)+y(0)=0.5+\ln(2).$$

Exact solution: $y(x)=\ln(1+x)$

Source: Nasir *et al.* (2018).

4.1. Region of Absolute Stability

The absolute stability region of the newly constructed block hybrid Nyström –type method (8) – (13) is plotted using the boundary locus method.

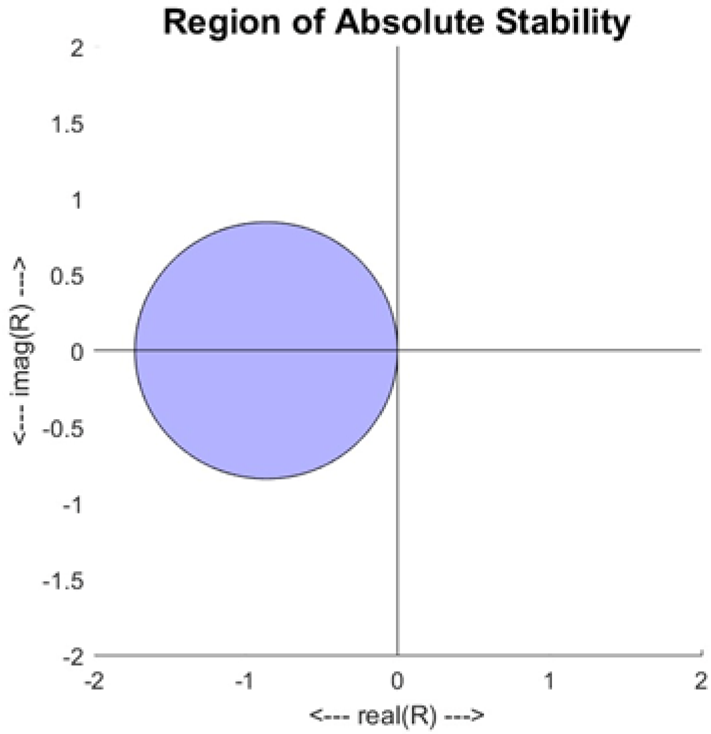

Figure 4.1: Region of Absolute Stability for BHNTM with five hybrid points.

Table 2: Comparison of Numerical result for Bratu problem with Impedance condition with the Proposed $HBNTM_{1,5}$ when $\lambda = 1$.

x	Exact solution	Phang *et al.* (2011)	Majid *et al.* (2011)	Nasir *et al.*, (2018)	Oguniran *et al.* (2023)	$BHNTM_{1,5}$
0.0	0.00	2.07×10^{-06}	1.18×10^{-05}	0.00	0.00	1.12×10^{-14}
0.1	0.18232155679	1.77×10^{-06}	1.25×10^{-05}	5.30×10^{-07}	1.25×10^{-13}	4.98×10^{-15}
0.2	0.26236424447	1.62×10^{-06}	1.36×10^{-05}	9.48×10^{-07}	2.33×10^{-13}	2.23×10^{-15}
0.3	0.33647223662	2.03×10^{-06}	1.46×10^{-05}	1.66×10^{-06}	7.38×10^{-13}	9.61×10^{-16}
0.4	0.40546510811	2.36×10^{-06}	1.47×10^{-05}	1.33×10^{-06}	5.66×10^{-13}	3.43×10^{-16}
0.5	0.47000362925	2.39×10^{-06}	1.44×10^{-05}	1.53×10^{-06}	2.96×10^{-13}	3.29×10^{-17}
0.6	0.53062825106	2.39×10^{-06}	1.37×10^{-05}	1.34×10^{-06}	6.81×10^{-13}	1.24×10^{-16}
0.7	0.58778666490	2.29×10^{-06}	1.29×10^{-05}	1.38×10^{-06}	8.66×10^{-13}	2.20×10^{-16}
0.8	0.64185388617	2.18×10^{-06}	1.20×10^{-05}	1.25×10^{-06}	1.23×10^{-13}	2.38×10^{-16}
0.9	0.69314718056	2.03×10^{-06}	1.10×10^{-05}	1.23×10^{-06}	5.48×10^{-12}	2.51×10^{-16}
1.0	0.09531017980	1.88×10^{-06}	1.01×10^{-05}	1.14×10^{-06}	9.30×10^{-12}	2.52×10^{-16}

Discussion or results

This research study aims to develop an implicitly continuous k-step Block Hybrid Nyström-Type Method using the power series polynomial with five hybrid points generated via the Bhaskara Cosine approximation formula to solve the Bratu problem with impedance boundary conditions. In order to access the accuracy of the proposed method, the present numerical results were compared with methods presented by Nasir *et al*. (2018) and Oguniran *et al*. (2023).

Table 2 shows clearly that the proposed ($BHNTM_{1,5}$) method requires no initial guess as recorded by Nasir *et al*. (2018) for solving example 1. Thus, as a result of this efficient performance, the proposed ($BHNTM_{1,5}$) method outperforms the existing methods found in literature in terms of error and accuracy and would rather be preferred in applications.

Conclusion

This study presents a single-step implicitly continuous Block Hybrid Nyström-Type Method of order seven to solve directly the nonlinear second order Bratu problem with impedance boundary conditions. The BHNTM are applied as simultaneous numerical integrator over non-overlapping subintervals and hence they present more accurate results when compared to methods found in literature. The method proposed is A(alpha) stable which makes the methods suitable for solving any kind of two-point boundary value problems with mixed boundary condition. However, future research will be to adopt the method to solve other forms of the two-point boundary value problems with mixed boundary conditions. Overview from the result showed a significant finding that the proposed Hybrid Block Nyström -Type Method gives an efficient and faster time of execution with better accuracy than the other methods found in literature. The proposed method outperforms the existing methods and can handle any kind of Impedance boundary conditions problems.

Declaration of competing interest

The authors declare that they have no known competing interests that could have appeared to influence the work reported in this paper.

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