LYAPUNOV FUNCTION FOR THE STABILITY OF FRACTIONAL DIFFERENTIAL-INTEGRAL RIEMANN-LIOUVILLE

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Abstract

In this paper the stability by using the direct method for some Lyapunov functions was applied to the system of fractional-differential-integral Riemann- Liouville type and the results explain the role of the Lyapunov in achieving stability. Some examples have been given to support the results.

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Introduction:

Stability is the first question in the theory of dynamic systems, and the study of stability worked for the first time in mechanics, because of the urgent need to study the equilibrium of systems and raise some questions about the motives of stability to introduce new mathematical concepts in general geometry, especially control engineering.

The theory of stability was of great importance to Scientists specializing in mathematics and scientists specializing in astronomy for a long time from the years and had a catalyst effect in these fields as the beginning was in problem to prove that the solar system is stable and the issue of analyzing the stability of differential equations still attracts the attention of many specialists despite its long history so the theory of stability in the modern era has become widely used in physics, astronomy, chemistry and even in biology.

In 1892, the Russian scientist Lyapunov published his doctoral thesis entitled The General Question of Movement Stability, which included many fruitful ideas and important results.

With these results, it became possible to divide the history of stability into two periods, the first before Lyapunov and the second after it, as it provided a precise definition of the stability of the movement as well as providing the two basic methods for analyzing stability issues.

exponentially stable of composite fractional Riemann-Liouville differential-integral system.

$$^R D^{\alpha_1}\, ^R D^{\alpha_2} z(v) = Az(v) + f(t, z(v),\, ^R D^{\gamma_1} z(v))\, \tilde{g}(t, z(v), I^{\gamma_2} z(v)) \tag{1.1}$$

$$^R D^{\alpha_1+\alpha_2-1} z(v)\,|_{t=0} = x_0$$

$$^R D^{\alpha_1+\alpha_2-2} z(v)\,|_{t=0} = \tilde{x}_0$$

$$^R D^{\alpha_2-1} z(v)\,|_{t=0} = \tilde{\tilde{x}}_0$$

Remark 1.1.

$$^{RL}_0 D^{\alpha}_t (^{RL}_0 D^{\beta}_t f(t)) = ^{RL}_0 D^{\alpha+\beta}_t f(t) - \sum_{j=1}^{m} \left[^{RL}_0 D^{\beta-j}_t f(t) \right]_{t=0} \frac{t^{-\alpha-j}}{\Gamma(1-\alpha-j)}$$

Where $n-1 \leq \alpha < n,\ m-1 \leq \beta < m$ and $m, n \in \mathbb{N}$

Lemma 1.1. Let $x = 0$ be an equilibrium point for the system (1.1), and

$D \subset \mathbb{R}^n$ be a domain including the origin.

Assume $v(v, x(v)): [0.\infty) \times D \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz such that

(i) $\quad \zeta_1 \|x\|^c \leq v(v, x(v)) \leq \zeta_2 \|x\|^{cd}$ (1.2)

(ii) $\quad ^R D^{\alpha}\, ^R D^{\beta} v(v, x(v)) \leq -\zeta_3 \|x\|^{cd}$ (1.3)

In which $v \geq 0, x \in D,\ \alpha, \beta \in (0,1), \zeta_1, \zeta_2, \zeta_3, c$ and d are arbitrary positive constants.

Then $z = 0$ is exponentially stable. If (i) (ii)holds globally on $\mathbb{R}^n$, then $x = 0$ is globally exponential stable.

<u>Proof:</u> from (1.2) and (1.3), we have:

$$^{R}D^{\alpha\,R}D^{\beta}\boldsymbol{v}(v,x(v)) = {}^{R}D^{\alpha+\beta}\boldsymbol{v}(v,x(v)) - \sum_{j=1}^{m-1}\left[{}^{R}D_{v}^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{t=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)} \leq -\zeta_{3}\|x\|^{cd}$$

$$^{R}D^{\alpha+\beta}\boldsymbol{v}(v,x(v)) \leq -\zeta_{3}\|x\|^{cd} + \sum_{j=1}^{m-1}\left[{}^{R}D_{v}^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}$$

$$^{R}D^{\alpha+\beta}\boldsymbol{v}(v,x(v)) \leq -\frac{\zeta_{3}}{\zeta_{2}}\boldsymbol{v}(v,x(v)) + \sum_{j=1}^{m-1}\left[{}^{R}D_{v}^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}$$

$$^{R}D^{\alpha+\beta}\boldsymbol{v}(v,x(v)) + K(v)$$

$$= -\frac{\zeta_{3}}{\zeta_{2}}\boldsymbol{v}(v,x(v)) + \sum_{j=1}^{m-1}\left[{}^{R}D_{v}^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)} \tag{1.4}$$

By Laplace transform for both sides of (1.4) we get

$$\mathcal{L}\left[{}^{R}D^{\alpha+\beta}\boldsymbol{v}(v,x(v)) + K(v)\right]$$

$$= \mathcal{L}\left[-\frac{\zeta_{3}}{\zeta_{2}}\boldsymbol{v}(v,x(v)) + \sum_{j=1}^{m-1}\left[{}^{R}D_{v}^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}\right]$$

$$\mathcal{L}\left[{}^{R}D^{\alpha+\beta}\boldsymbol{v}(v,x(v))\right] + \mathcal{L}[K(v)]$$

$$= -\frac{\zeta_{3}}{\zeta_{2}}\mathcal{L}[\boldsymbol{v}(v,x(v))] + \mathcal{L}\left[\sum_{j=1}^{m-1}\left[{}^{R}D_{v}^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}\right]$$

$$s^{\alpha+\beta}\mathcal{L}[\boldsymbol{v}(v,x(v))] - \sum_{j=0}^{n-1}s^{j}\left[{}^{R}D_{v}^{\alpha-j-1}\boldsymbol{v}(v,x(v))\right]_{v=0} + \mathcal{L}[K(v)] = -\frac{\zeta_{3}}{\zeta_{2}}\mathcal{L}[\boldsymbol{v}(v,x(v))]$$

$$+ \mathcal{L}\left[\sum_{j=1}^{n-1}\left[{}^{R}D_{v}^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}\right]$$

$$\left(s^{\alpha+\beta} + \frac{\zeta_3}{\zeta_2}\right)\mathcal{L}[\boldsymbol{v}(v,x(v))]$$

$$= \sum_{j=0}^{n-1} s^j \left[{}^{R}D_v^{\alpha-j-1}\boldsymbol{v}(v,x(v))\right]_{v=0} - \mathcal{L}[k(v)] + \mathcal{L}\left[\sum_{j=1}^{m-1}\left[{}^{R}D_v^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}\right]$$

Let $\qquad N := N(s,\zeta) = \left(s^{\alpha+\beta} + \frac{\zeta_3}{\zeta_2}\right) N \ \boldsymbol{v}(s) = \sum_{j=0}^{n-1} s^j \left[{}^{R}D_v^{\alpha-j-1}\boldsymbol{v}(v,x(v))\right]_{v=0} - K(s) +$

$\mathcal{L}\left[\sum_{j=1}^{m-1}\left[{}^{R}D_v^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}\right]$

$$\boldsymbol{v}(s) = \frac{1}{N}\left[\sum_{j=0}^{n-1} s^j \left[{}^{R}D_v^{\alpha-j-1}\boldsymbol{v}(v,x(v))\right]_{v=0}\right] - \frac{1}{N}K(s)$$

$$+\frac{1}{N}\mathcal{L}\left[\sum_{j=1}^{m-1}\left[{}^{R}D_v^{\beta-j}\boldsymbol{v}(v,x(v))\right]_{v=0}\frac{v^{-\alpha-j}}{\Gamma(1-\alpha-j)}\right]$$

If $\ {}^{R}D_v^{\alpha-j-1}\boldsymbol{v}(v,x(v))|_{v=0} = 0 \qquad$ and $\qquad {}^{R}D_v^{\beta-j}\boldsymbol{v}(v,x(v))|_{v=0} = 0 \ $ then

The solution (1.1) is a zero solution

If $\ {}^{R}D_v^{\alpha-j-1}\boldsymbol{v}(v,x(v)|_{v=0} \neq 0 \qquad$ and $\qquad {}^{R}D_v^{\beta-j}\boldsymbol{v}(v,x(v))|_{v=0} \neq 0 \ $ then

$$\boldsymbol{v}(s) = \sum_{j=0}^{n-1}\frac{a_j s^j}{N} - \frac{K(s)}{N} + \sum_{j=1}^{m-1}\frac{b_j s^{\alpha+j-1}}{N}$$

$$\boldsymbol{v}(s) = \sum_{j=0}^{n-1}\frac{a_j s^j}{N} + \sum_{j=1}^{m-1} b_j \frac{s^{\alpha+j-1}}{N} - \frac{K(s)}{N}$$

Now, we have to prove $\mathcal{L}^{-1}\left\{\frac{K(s)}{N}\right\}$ is nonnegative

$$\mathcal{L}^{-1}\left\{\frac{K(s)}{N}\right\} = \mathcal{L}^{-1}\left\{\frac{K(s)}{s^{\alpha+\beta} + \frac{\zeta_3}{\zeta_2}}\right\} = \mathcal{L}^{-1}\left\{\frac{K(s)}{\frac{\zeta_2 s^{\alpha+\beta} + \zeta_3}{\zeta_2}}\right\} = \mathcal{L}^{-1}\left\{\frac{K(s)\zeta_2}{\zeta_2 s^{\alpha+\beta} + \zeta_3}\right\}$$

$$= \zeta_2 \mathcal{L}^{-1}\left\{\frac{K(s)}{s^{\alpha+\beta} + \zeta_2}\right\} = \zeta_2 \mathcal{L}^{-1}\left\{\frac{K(s-\zeta_2)}{s^{\alpha+\beta}}\right\} = \zeta_2 \mathcal{L}^{-1}\left\{\frac{\mathcal{L}(e^{\zeta_2}K(v))}{s^{\alpha+\beta}}\right\}$$

$$= \zeta_2 e^{\zeta_2} \mathcal{L}^{-1} \left\{ \frac{\mathcal{L}(K(v))}{s^{\alpha+\beta}} \right\} \tag{1.5}$$

Using $\mathcal{L}^{-1}\{s^{\alpha+\beta}M(s)\} = m'(v) + m(0)$,

we have that $\mathcal{L}^{-1}\left\{ \frac{sK(s)}{s^{\alpha+\beta+\zeta_2}} \right\} = p'(v) + p(0)$

Where $p(v) = \mathcal{L}^{-1}\left\{ \frac{K(s)}{s^{\alpha+\beta+\zeta_2}} \right\} = e^{\zeta_2} \int_0^v K(z)dz \geq 0$ and $p(0) = 0$

Thus $p'(v) = e^{\zeta_2} K(v) \geq 0$

therefore (1.5) is nonnegative, so, we get $\mathcal{L}^{-1}\left\{ \frac{K(s)}{N} \right\}$ is nonnegative, we have that

$$\boldsymbol{v}(v, x(v)) \leq \mathcal{L}^{-1}\left(\sum_{j=0}^{n-1} \frac{a_j s^j}{N} \right) + \sum_{j=1}^{m-1} \frac{b_j v^{-\alpha-j}}{N\Gamma(1-\alpha-j)}$$

$$\boldsymbol{v}(v, x(v)) \leq \sum_{j=0}^{n-1} \frac{a_j}{v^j N} + \sum_{j=1}^{m-1} \frac{b_j v^{-\alpha-j}}{N\Gamma(1-\alpha-j)} \tag{1.6}$$

From (1.6) and (1.5), we get that

$$\|x(v)\| \leq \sum_{j=0}^{n-1} \frac{a_j}{\zeta_1 v^j N} + \sum_{j=1}^{m-1} \frac{b_j v^{-\alpha-j}}{\zeta_1 N\Gamma(1-\alpha-j)}$$

Thus, then $x(v)$ is exponentially stable.

Theorem 1.1. Let $\boldsymbol{v}(v, x(v)): \Omega \longrightarrow R$ and $x(v): [v\_0, \infty] \longrightarrow \Omega$ are continuous differentiable functions, $\Omega \subset \mathbb{R}^n$ and $\boldsymbol{v}(v, x(v))$ is convex over Ω . Then

$$\prescript{R}{v_0}{D}_v^{\zeta_1} \prescript{R}{v_0}{D}_v^{\zeta_2} \boldsymbol{v}(x(v)) \leq \left(\frac{\partial \boldsymbol{v}}{\partial x} \right)^T \prescript{R}{v_0}{D}_v^{\zeta_1} \prescript{R}{v_0}{D}_v^{\zeta_2} x(v), \qquad \forall \zeta_1, \zeta_2 \in (0,1) \tag{1.7}$$

Proof. From (1.7), we have that

$$\prescript{R}{v_0}{D}_v^{\zeta_1} \prescript{R}{v_0}{D}_v^{\zeta_2} \boldsymbol{v}(x(v)) - \left(\frac{\partial \boldsymbol{v}}{\partial x} \right)^T \prescript{R}{v_0}{D}_v^{\zeta_1} \prescript{R}{v_0}{D}_v^{\zeta_2} x(v) \leq 0 \tag{1.8}$$

$$\begin{aligned}
&{}_{v_0}^{R}D_v^{\zeta_1+\zeta_2}\boldsymbol{v}\big(x(v)\big) - \sum_{j=1}^{m}\big[{}_{v_0}^{R}D_v^{\zeta_2-j}\boldsymbol{v}(x(v))\ \big|_{v=0}\frac{\boldsymbol{v}^{-\zeta_1-j}}{\Gamma(1-\zeta_1-j)} \\
&\qquad -\left(\frac{\partial \boldsymbol{v}}{\partial x}\right)\left({}_{v_0}^{R}D_v^{\zeta_1+\zeta_2}x(v) - \sum_{j=1}^{m}\big[{}_{v_0}^{R}D_v^{\zeta_2-j}x(v)\ \big|_{v=0}\frac{\boldsymbol{v}^{-\zeta_1-j}}{\Gamma(1-\zeta_1-j)}\right)
\end{aligned}$$

$$= {}_{v_0}^{c}D_v^{\zeta_1+\zeta_2}\boldsymbol{v}\big(x(v)\big) + \sum_{j=0}^{n-1}\boldsymbol{v}(0)\frac{\boldsymbol{v}^{1-(\zeta_1+\zeta_2)}}{\Gamma(1+j-(\zeta_1+\zeta_2))}$$

$$-\sum_{j=1}^{m}\big[{}_{v_0}^{R}D_v^{\zeta_2-j}\boldsymbol{v}(x(v))\ \big|_{v=0}\frac{v^{-\zeta_1-j}}{\Gamma(1-\zeta_1-j)} - \left(\frac{\partial \boldsymbol{v}}{\partial x}\right)\ ,\quad n=[\zeta_1+\zeta_2]+1$$

$$\left({}_{v_0}^{c}D_v^{\zeta_1+\zeta_2}x(v) + \sum_{j=0}^{n-1}v^{j}(0)\frac{v^{1-(\zeta_1+\zeta_2)}}{\Gamma(1+j-(\zeta_1+\zeta_2))} - \sum_{j=1}^{m}\big[{}_{v_0}^{R}D_v^{\zeta_2-j}x(v)\ \big|_{v=0}\frac{V^{-\zeta_1-j}}{\Gamma(1-\zeta_1-j)}\right)$$

$$\frac{(\zeta_1+\zeta_2)M(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}\int_{t_0}^{t}\boldsymbol{v}'(x(s))\ exp\left[-\frac{(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}(v-s)\right]ds$$

$$+\sum_{j=0}^{n-1}v^{j}(0)\frac{\boldsymbol{v}^{1-(\zeta_1+\zeta_2)}}{\Gamma(1+j-(\zeta_1+\zeta_2))} - \sum_{j=1}^{m}\big[{}_{v_0}^{R}D_v^{\zeta_2-j}\boldsymbol{v}(x(v))\ \big|_{v=0}\frac{\boldsymbol{v}^{-\zeta_1-j}}{\Gamma(1-\zeta_1-j)}$$

$$-\left(\frac{\partial \boldsymbol{v}}{\partial x}\right)\left(\frac{(\zeta_1+\zeta_2)M(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}\int_{v_0}^{v}x'(s)exp\left[\frac{-(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}(t-s)\right]d\right.$$

$$\left.+\sum_{j=0}^{n-1}\boldsymbol{v}^{j}(0)\frac{\boldsymbol{v}^{1-(\zeta_1+\zeta_2)}}{\Gamma(1+j-(\zeta_1+\zeta_2))} - \sum_{j=1}^{m}\big[{}_{v_0}^{R}D_v^{\zeta_2-j}x(v)\ \big|_{v=0}\frac{\boldsymbol{v}^{-\zeta_1-j}}{\Gamma(1-\zeta_1-j)}\right) \le 0$$

If $x(0)=0$, $\boldsymbol{v}(x(0))=0$, ${}_{v_0}^{R}D_v^{\zeta_2-j}x(v)\ |_{v=0}=0$

and ${}_{v_0}^{R}D_v^{\zeta_2-j}\boldsymbol{v}(x(v))\ |_{v=0}=0$

Then,

$$\frac{(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}\int_{v_0}^{v}\left(\frac{\partial v(x(s))}{\partial z}-\frac{\partial v(x(v))}{\partial v}\right)\cdot x'(s)\ exp\left[-\frac{(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}(v-s)\right]ds \le 0 \qquad (1.9)$$

Suppose that $\Phi(s,v) = \boldsymbol{v}\big(x(s)\big) - \boldsymbol{v}\big(x(v)\big) - \left(\frac{\partial v}{\partial x}\right)^T (x(s) - x(v))$

Then (1.9) become to

$$\frac{(\zeta_1+\zeta_2)M(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)} \int_{v_0}^{v} \frac{d}{ds}\Phi(s,v)\, exp\left[-\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}(v-s)\right] ds$$

$$= \frac{(\zeta_1+\zeta_2)M(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)} \int_{v_0}^{v} d[\Phi(s,v)]\, exp\left[-\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}(v-s)\right] ds \le 0$$

Thus

$$\frac{(\zeta_1+\zeta_2)M(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)} \int_{v_0}^{v} d[\Phi(s,v)]\, exp\left[-\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}(v-s)\right] ds \qquad (1.10)$$

$$= \frac{(\zeta_1+\zeta_2)M(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}\left[-exp\left[-\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}(v-v_0)\right]\Phi(v_0,v)\right.$$
$$\left. -\frac{\zeta}{1-\alpha}\int_{v_0}^{v}\Phi(s,v)\, exp\left[-\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}(v-s)\right] ds\right]$$

From the convexity of $\boldsymbol{v}(v, x(v))$, we get $\Phi(s,v) \ge 0$

Thus

$$\frac{(\zeta_1+\zeta_2)M(\zeta_1+\zeta_2)}{1-(\zeta_1+\zeta_2)}\left[-exp\left[-\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}(v-v_0)\right]\Phi(0,v)\right.$$
$$\left. -\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}\int_{v_0}^{v}\Phi(s,v)\, exp\left[-\frac{\zeta_1+\zeta_2}{1-(\zeta_1+\zeta_2)}(v-s)\right] ds\right] \le 0$$

The following result gives the interesting result of the globally exponential stable for differential integral fractional system

Theorem 1.2. Let $x = 0$ be the equilibrium point of (1.1) suppose that there exists a Lyapunov function $\boldsymbol{v}(v, x(t))$ and locally Lipchitz on X, such that

$$\gamma_1 \|x\|^c \leq \boldsymbol{v}(v, x(v)) \leq \gamma_2 \|x\|^c$$

$$\left(\frac{\partial \boldsymbol{v}}{\partial x}\right)^T (Ax(v)) + f(t, x(v), {}^R D^{\gamma_1} x(v)) \leq -\gamma_3 \|x\|^{cd}$$

where $\gamma_1, \gamma_2, \gamma_3, c$ and d are arbitrary positive constants. Then, $x = 0$ is a globally exponential stable.

Proof. By theorem (1.1), we get

$$ {}^R_{v_0}D^{\zeta_1}_v {}^R_{v_0}D^{\zeta_2}_v \boldsymbol{v}(x(v)) \leq \left(\frac{\partial \boldsymbol{v}}{\partial x}\right)^T {}^R_{v_0}D^{\zeta_1}_v {}^R_{v_0}D^{\zeta_2}_v x(v)$$

$$ = \left(\frac{\partial \boldsymbol{v}}{\partial x}\right)^T (Ax(v)) + f(t, x(v), {}^R D^{\gamma_1} x(v)) \leq -\gamma_3 \|x\|^{cd}$$

By lemma (1.1), $x = 0$ is globally exponentially stable.

Lemma 1.2. Let $x(0) = u(0)$ and ${}^R_{v_0}D^{\zeta_1} {}^R_{v_0}D^{\zeta_2}_v x(v) \geq {}^R_{v_0}D^{\zeta_1} {}^R_{v_0}D^{\zeta_2} u(v)$

Where $\zeta_1 + \zeta_2 \in (0.1)$ then $x(u) \geq u(v)$

Proof. In fact ${}^R_{v_0}D^{\zeta_1} {}^R_{v_0}D^{\zeta_2} x(v) = \psi(v) + {}^R_{v_0}D^{\zeta_1} {}^R_{v_0}D^{\zeta_2} u(v)$

By using the fractional Laplace transform operator, we get

$$\mathcal{L}({}^R_{v_0}D^{\zeta_1} {}^R_{v_0}D^{\zeta_2} x(v))$$

$$ = \mathcal{L}(\Psi(v)) + \mathcal{L}\left({}^R_{v_0}D^{\zeta_1} {}^R_{v_0}D^{\zeta_2} u(v)\right), \qquad \zeta_1 + \zeta_2 < 1$$

$$ s^{\zeta_1+\zeta_2} x(s) = \Psi(v) + s^{\zeta_1+\zeta_2} U(s)$$

$$z(s) = Lx(v), \quad k(s) = L\Psi(v), \quad U(s) = Lu(v),$$

we get $\quad x(s) = \dfrac{\Psi(s)}{s^{\zeta_1 + \zeta_2}} + U(s)$

By Laplace inverse and by $\Psi(v) \geq 0$, we get

$$x(v) = {}^R D^{\zeta_1 + \zeta_2}\Psi(v) + u(v), \quad x(v) \geq u(v)$$

Theorem 1.3. Let $x = 0$ be an equilibrium point for the non-autonomous composite fractional-order system (1.1) with Lyapunov function $v(v, x(v))$ and class K-function γ_i $(i = 1,2,3)$ such that

$$\gamma_1(\|x\|) \leq v(v, x(v)) \leq -\gamma_2(\|x\|) \tag{1.11}$$
$$ {}_{v_0}^R D^{\zeta_1} {}_{v_0}^R D^{\zeta_2} v(v, x(v)) \leq -\gamma_3(\|x\|) \tag{1.12}$$

Where $\zeta_1 + \zeta_2 \in (0,1)$ then $x = 0$ is asymptotically stable.

Proof. By (1.11) and (1.12), we get

$${}_{v_0}^R D^{\zeta_1} {}_{v_0}^R D^{\zeta_2} v(v, x(v)) \leq -\gamma_2(\gamma_2^{-1}(v(v, x(v)))) \tag{1.13}$$

By lemma (1.1) implies that $v(v, x(v)) \geq 0$ then $v(t, x(t)) \leq v(0, x(0))$

Now we have two cases

1.   For $\epsilon > 0$ such that $v(v, x) \geq \epsilon$, $v \geq 0$, $0 < \epsilon < v(v, x) \leq v(0, z(0))$, for $v \geq 0$
By (1.11), and (1.12) we have

$${}_{v_0}^R D^{\zeta_1} {}_{v_0}^R D^{\zeta_2} v(v, x(v)) \leq -\gamma_3(\gamma_2^{-1}(v(v, x(v)))) \leq -\lambda v(v, x(v))$$

where $= \dfrac{\gamma_3(\gamma_2^{-1}\epsilon)}{v(0, x(0))} > 0$, then $v(v, x(v)) \leq \dfrac{v(0, z(0))}{2 - \zeta_1 + \zeta_2} e^{\frac{(\zeta_1 + \zeta_2)v}{2 - \zeta_1 + \zeta_2}}$

2.   Let $v_1 \geq 0$ and satisfying $v(v, x(v)) = 0$
By (1.11), we get $z(v_1) = 0$ and then $x(v) = 0$ for $v \geq v_1$

From (1) which contradicts which hypothesis states that $v(v, x(v)) \geq \epsilon$

Thus from (1) and (2) we get

$$\lim_{v \to \infty} v(v, x(v)) = 0 \text{ and we get } \lim_{v \to \infty} x(v) = 0$$

Lemma 1.3. For $m < \alpha < m + 1$, $m \in \mathbb{N}_0$, and $v \in C^m[0, T]$, the following conditions are equivalent:

i)   the fractional derivative $^c D^\alpha v \in C[0, T]$ exists;

ii)   a finite limit $\lim_{t \to 0} t^{m-\alpha} \left(v^{(m)}(t) - v^{(m)}(0) \right) =: \gamma_m$ exists, and

$$\sup_{0 < t \leq T} \left| \int_{\theta t}^{t} (t - s)^{m-\alpha-1} \left(v^{(m)}(t) - v^{(m)}(s) \right) ds \right| \to 0 \text{ as } \theta \uparrow 1;$$

iii) $v^{(m)}$ has the structure $v^{(m)} - v^{(m)}(0) = \gamma_m t^{\alpha-m} + v_m$ where γ_m is a constant, $v_m \in \mathcal{H}_0^{\alpha-m}[0, T]$,

and $\int_0^t (t - s)^{m-\alpha-1} \left(v^{(m)}(t) - v^{(m)}(s) \right) ds =: w_m(t)$ converges for every

$t \in (0, T]$ defining a function $w_m \in C(0, T]$ which has a finite limit

$$\lim_{t \to 0} w_m(t) =: w_m(0)$$

For $v \in C^m[0, T]$ with $D_{Cap}^\alpha v \in C[0, T]$, it holds $(^c D^\alpha v)(0) = \Gamma(\alpha + 1 - m)\gamma_m$,

$$(^c D^\alpha v)(t) = \frac{1}{\Gamma(m + 1 - \alpha)} \left(t^{m-\alpha} \left(v^{(m)}(t) - v^{(m)}(0) \right) \right.$$

$$\left. + (\alpha - m) \int_0^t (t - s)^{m-\alpha-1} \left(v^{(m)}(t) - v^{(m)}(s) \right) ds \right), \qquad 0 < t \leq T .$$

Theorem 1.4. Let $v \in C([0, T], \mathbb{R}^n)$, $\quad \alpha \in (0,1)$, the following conditions (i), (ii) are equivalent

(i)   The composite fractional derivative
$^R D_{0+}^{\alpha_1} {}^R D_{0+}^{\alpha_2} v \in C([0, T], \mathbb{R}^n)$ exists,

(ii) $\lim\limits_{t\to 0} \dfrac{v(t)-v(0)}{t^\alpha} = \gamma$ exists,

and

$\sup\limits_{0<t\le T} \left\| \int_{\theta t}^t (v(t)-v(\tau))(t-\tau)^{-\alpha}d\tau \right\| \to 0$ as $\theta \to 1$

$$^R D_{0+}^\alpha v(t) = {}_0^c D_t^\alpha f(t) + \sum_{j=0}^{n-1} \frac{v(0)^{(j)} t^{(j-\alpha)}}{\Gamma(1+j-\alpha)}$$

$$= \frac{v(t)-v(0)}{\Gamma(1-\alpha)t^{-\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t (v(t)-v(\tau))(t-\tau)^{-\alpha}d\tau \ , \quad n-1<\alpha, 0<t\le T$$

$v \in C([0,T],\mathbb{R}^n)$ and $^R D_{0+}^\alpha v \in C([0,T],\mathbb{R}^n), {}^R D_{0+}^\alpha(0) = \Gamma(\alpha+1)\gamma$

For $v \in C([0,T],\mathbb{R}^n)$ having fractional derivations.

Proof. By lemma (1.3), we have that, $D \subset \mathbb{R}^n$ is an open set and $0 \in D$, by considering the following equation.

$$^R D^{\alpha_1}{}^R D^{\alpha_2} x(v) = Ax(v) + f(t,x(v),{}^R D^{\gamma_1}x(v))g(t,x(v),I^{\gamma_2}x(v)) \tag{1.14}$$

Where $f: R^+ \times D \times D \to \mathbb{R}^n$ satisfies the following

i)$f(t,0,0)g(t,0) = 0$

ii) $f(t,\cdot,\cdot)\tilde{g}(t,\cdot)$ locally Lipchitz continuous in Nbh (0) by lemma (1.3), there exists a solution with for system (1.1)

Let $\varphi: I \times \mathbb{R}^n \to \mathbb{R}^n, \quad t \to \varphi(t,x_0,\tilde{x}_0,\tilde{x}_0)$ the solution denoted of the system (1.1) on maximal interval of $I = [0, t_{max}(x_\circ, \tilde{x}_\circ, \tilde{x}_\circ)$ with $0 < t_{max}(x_\circ, \tilde{x}_\circ, \tilde{x}_\circ) \le \infty$

The following there interesting for using the Lyapunov direct method to give an asymptotically behavior solution

Theorem 1.5. For a given $x_0 \in \mathbb{R}^n$, and $V: \mathbb{R}^n \to \mathbb{R}$ satisfied, the following

i)The function V definition on $\mathbb{R}^n$ convex function and $v(0) = 0$

ii)   The function V defined on $\mathbb{R}^n$ and differentiable

Then

$$^R D_{0+}^{\alpha_1}{}^R D_{0+}^{\alpha_2} V(u(t)) \le \langle \nabla V(u(t)), {}^R D_{0+}^{\alpha_1}{}^R D_{0+}^{\alpha_2} u(t) \rangle \tag{1.15}$$

Where ∇V is the gradient of the function V.

Proof. Form $u \in \{x_0\} + I_{0+}^{\alpha_1} I_{0+}^{\alpha_2} C([0,T],\mathbb{R}^n)$ thus, $\psi \in C([0,T],\mathbb{R}^n)$ such that

$u = x_0 + I_{0+}^{\alpha_1} I_{0+}^{\alpha_2} \psi$ exist and continuous on $[0, T]$, also from

$${}^R D_{0+}^{\alpha_2} {}^R D_{0+}^{\alpha_1} u(0) = c \frac{t^{-(\alpha_1, -\alpha_2)}}{\Gamma(1-(\alpha_1+\alpha_2))} \tag{1.16}$$

$${}^R D_{0+}^{\alpha_1} {}^R D_{0+}^{\alpha_2} u(t) = \frac{u(t) - u(0)}{\Gamma(1-\alpha)} \frac{d}{dt} t^\alpha + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t u(t) - u(\tau)(t-\tau)^{-\alpha} d\tau$$

$,0 < t \leq T$ (1.17)

Using (i) (ii), and $\lim_{t\to 0} \left(V(u(t)) - V(u(0)) \right) t^{-\alpha} = (\nabla V(u(0)), \gamma)$ and from (ii), we get

$$\sup_{0 < t \leq T} \left\| \int_{\theta t}^t V(u(t)) - V(u(\tau))(t-\tau)^{-\alpha} d\tau \right\| \to 0$$

And from theorem (1.4) show that

$${}^R D_{0+}^{\alpha_2} {}^R D_{0+}^{\alpha_1} V(u(0)) = c \frac{t^{-(\alpha_1, -\alpha_2)}}{\Gamma(1+(\alpha_1+\alpha_2))} \left(\nabla V(u(0)), \gamma \right) \quad \text{for } t \in (0, T]. \tag{1.18}$$

$${}^R D_{0+}^{\alpha_2} {}^R D_{0+}^{\alpha_1} V(u(t)) = \frac{V(u(t)) - V(u(0))t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t V(u(t)) - V(u(\tau))(t-\tau)^{-\alpha} d\tau$$

(1.19)

From (1.16) and (1.18), we have

$${}^R D_{0+}^{\alpha_1} {}^R D_{0+}^{\alpha_2} V(u(0)) = \langle \nabla V(u(0)), {}^R D_{0+}^{\alpha_1} {}^R D_{0+}^{\alpha_2} u(0) \rangle \tag{1.20}$$

For $0 < t \leq T$ using (1.17) and (1.19), we get

$$^{R}D_{0+}^{\alpha_1}{}^{R}D_{0+}^{\alpha_2}V\big(u(t)\big) - \langle \nabla V\big(u(t)\big), {}^{R}D_{0+}^{\alpha_1}{}^{R}D_{0+}^{\alpha_2}u(t)\rangle$$

$$= \frac{(t)^{-\alpha}V\big(u(t)\big) - V\big(u(0)\big) - \langle \nabla V\big(u(t)\big), u(t) - u(0)\rangle}{\Gamma(1-\alpha)}$$

$$+ \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{t} V\big(u(t)\big) - V\big(u(\tau)\big)(t-\tau)^{-\alpha}d\tau - \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{t} \langle \nabla V\big(u(t)\big), u(t) - u(\tau)\rangle$$

$$> (t-\tau)^{-\alpha}d\tau$$

(1.21)

Because V is convex and differentiable, we obtain

$$V\big(u(t)\big) - V\big(u(\tau)\big) - \langle \nabla V\big(u(t)\big), u(t) - u(\tau)\rangle \leq 0 \quad \text{for } 0 \leq \tau \leq t \leq T \text{ with } (1.20), \text{ and } (1.21)$$

We get

$$^{R}D_{0+}^{\alpha_1}{}^{R}D_{0+}^{\alpha_2}V\big(u(t)\big) \leq \langle \nabla V\big(u(t)\big), {}^{R}D_{0+}^{\alpha_1}{}^{R}D_{0+}^{\alpha_2}u(t)\rangle, \forall\, t \in [0,T]$$

The following results are very interesting for an asymptotically stable nonlinear intgro fractional system

Theorem 1.6. Consider the following composite fractional equation

$$^{R}D_{0+}^{\alpha_1}{}^{R}D_{0+}^{\alpha_2}x(t) = f\big(t, x(t), D^{\gamma_1}x(t)\big)\tilde{g}\big(t, x(t), I^{\gamma_2}x(t)\big)$$

and $f: D \times R^n \times R^n$ satisfies $f(t,0,0)\tilde{g}(t,0,0) = 0$, and f, g is local Lipchitz continuous in the neighborhood of origin. Let $V: \mathbb{R}^n \to \mathbb{R}^+$ satisfying

i) The function V is convex and differentiable on $\mathbb{R}^+$
ii)   There exist constants $a, b, C_1, C_2, r > 0$ such that
$C_1\|x\|^a \leq V(x) \leq C_2\|x\|^b$ for all $x \in B_r(0)$;
iii)   There are constants $C_3 \geq 0$ and $c \geq b$ such that
$\langle \nabla V(x), f\big(t, x(t), D^{\gamma_1}x(t)\big)\tilde{g}\big(t, x(t), I^{\gamma_2}x(t)\big)\rangle \leq -c_3\|x\|^c$ for all $x \in B_r(0)$ then
(a)   The trivial solution is stable if $C_3 = 0$;
(b)   The trivial solution is asymptotically stable if $C_3 > 0$ and $c \geq b$
Proof. By $f(t,0,0)\tilde{g}(t,0,0) = 0$ there is $r_1 \in (0, r)$ such that f is Lipschitz continuous on $B_{r_1}(0)$ and let L be a Lipschitz constant of f on $B_{r_1}(0)$ also let F be an extended Lipschitz function of f such that

$$F(t,x) = f\big(t, x(t), D^{\gamma_1}x(t)\big)\tilde{g}\big(t, x(t), I^{\gamma_2}x(t)\big), x \in B_{r_1}(0)$$

hence, for any $\varepsilon \in (0, r_1)$, set $\delta = \frac{1}{K}(\frac{c_1}{c_2})^{1/b}\varepsilon^{a/b}$, for $K > 1$ is large enough to $\delta < \varepsilon$.

Now for $x_0 \in B_\delta(0)$ the solution of

$$\begin{cases} {}^cD_{0+}^{\alpha_1}{}^cD_{0+}^{\alpha_2}x(t) = F(t, x(t)), t > 0 \\ \qquad\qquad x(0) = x_0 \end{cases} \tag{1.22}$$

Is $\tilde{\varphi}(., x_0)$ and it is unique on the interval $[0, \infty)$ and $\|\tilde{\varphi}(t, x_0)\| = \varepsilon$, put

$t_0 := \inf\{t > 0 : \|\tilde{\varphi}(t, x_0)\| \geq \varepsilon\}$, then $t_0 > 0$ and $\|\tilde{\varphi}(t_0, x_0)\| = \varepsilon$ and

$\|\tilde{\varphi}(t, x_0)\| < \varepsilon \; \forall t \in (0, t_0)$ then $\tilde{\varphi}$ satisfy (i), (ii).

(a)   Suppose that $C_3 = 0$ from theorem (1.5), we get
${}^RD_{0+}^{\alpha_1}{}^RD_{0+}^{\alpha_2}V(\tilde{\varphi}(t, x_0)) \leq \langle \nabla V(\tilde{\varphi}(t, x_0)), F(\tilde{\varphi}(t, x_0)) \rangle \leq 0$ for all $t \in [0, t_0]$ Then

$$V(\tilde{\varphi}(t, x_0)) \leq V(x_0) \; \forall t \in [0, t_0] \tag{1.23}$$ This
combining with (i), we get

$$\|\tilde{\varphi}(t, x_0)\| \leq \left(\frac{c_2}{c_1}\|x_0\|^b\right)^{1/a} \forall t \in [0, t_0]$$

Hence

$$\|\tilde{\varphi}(t_0, x_0)\| \leq \left(\frac{c_2}{c_1}\|x_0\|^b\right)^{1/a} \leq \left(\frac{c_2}{c_1}\delta^b\right)^{1/a} \tag{1.24}$$

It is contradiction
Thus $\|\tilde{\varphi}(t, x_0)\| < \varepsilon \; \forall t \in [0, \infty)$
hence $\tilde{\varphi}(., x_0)$ is a solution of the equation with $x(0) = x_0$ and it is stable.
(b)   Let $C_3 > 0$, as in (a), we have that $\tilde{\varphi}(t, x_0) = 0$ is a solution of (1.14) is stable.

For $\varepsilon > 0, \varepsilon < r_1, \exists \delta > 0$ such that the solution $\tilde{\varphi}(t, x_0)$ of (1.14) with $\|x_0\| < \delta$ satisfies $\|\tilde{\varphi}(t, x_0)\| < \varepsilon$, for all $t \geq 0$

From theorem (1.5) and from conditions (ii) (iii) we have that

$${}^RD_{0+}^{\alpha_1}{}^RD_{0+}^{\alpha_2}V(\tilde{\varphi}(t, x_0)) \leq -c_3\|\tilde{\varphi}(t, x_0)\|^c \leq -\frac{c_3}{c_2^{c/b}}(V(\tilde{\varphi}(t, x_0)))^{c/b} \quad \text{for all } t \geq 0.$$

Put $A := \frac{-c_3}{c_2^{c/b}}, p := \frac{c}{b}$ and consider the following

$$^{R}D_{0+}^{\alpha_1}{}^{R}D_{0+}^{\alpha_2}y(t) = Ay^p(t), t > 0 \qquad y(0) = y_0 > 0 \tag{1.25}$$

The solution $\Phi(\cdot, y_0)$ to (1.25) exists on the whole interval $[0, \infty)$ and satisfies

On the other hand, we have that

$$V\big(\tilde{\varphi}(t, x_0)\big) \leq \Phi\big(t, V(x_0)\big), \ \forall t \geq 0 \ \text{ thus, for any } \ x_0 \in B(0, \delta) \setminus \{0\}$$

We get $\qquad \lim_{t \to \infty} \|\tilde{\varphi}(t, x_0)\|^a \leq \dfrac{1}{C_1} \lim_{t \to \infty}$

$$V\big(\tilde{\varphi}(t, x_0)\big) \leq \frac{1}{C_1} \lim_{t \to \infty} \Phi\big(t, V(x_0)\big) = 0$$

From existence and uniqueness (1.22) if $x_0 = 0$ then $\tilde{\varphi}(\cdot, 0) = 0$, so the trivial solution of (1.14) is asymptotically stable.

Example 1.1. Consider the following compost fractional Riemann- Liouville derivative integral systems follows $^{R}D^{\alpha_1}{}^{R}D^{\alpha_2}x_1(v) = -x_1(v) - (x_2^{\frac{1}{2}}(v) + {}^{R}D^{\gamma_1}x_2)(x_2^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v))$

$$^{R}D^{\alpha_1}{}^{R}D^{\alpha_2}x_2(v) = -x_2(v) - (x_1^{\frac{1}{3}}(v) + {}^{R}D^{\gamma_1}x_1)(x_1^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v))$$

the Lyapunov function $V\big(v, x(v)\big) = x_1^{\frac{4}{3}}(v) + x_2^{\frac{4}{3}}(v)$

Where $(v) = [x_1(v), x_2(v)]^T$, we have $\left(\dfrac{\partial v}{\partial t}\right)^T \begin{bmatrix} -x_1(v) - (x_2^{\frac{1}{2}}(v) + {}^{R}D^{\gamma_1}x_2)(x_2^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v)) \\ x_2(v) - (x_2^{\frac{1}{3}}(v) + {}^{R}D^{\gamma_1}x_1)(x_1^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v)) \end{bmatrix}$

$$\begin{cases} \left(\dfrac{\partial v}{\partial x}\right)^T \begin{bmatrix} -x_1(v) - (x_2^{\frac{1}{2}}(v) + {}^{R}D^{\gamma_1}x_2)(x_2^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v)) \\ x_2(v) - (x_2^{\frac{1}{3}}(v) + {}^{R}D^{\gamma_1}x_1)(x_1^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v)) \end{bmatrix} \leq -V(v, x(v)) \\ \qquad \|x(v)\|^{\frac{4}{3}} \leq V(v, x(v)) \leq \sqrt[3]{2}\|x(v)\|^{\frac{4}{3}} \end{cases}$$

Then $\quad \left(\dfrac{\partial v}{\partial x}\right)^T \begin{bmatrix} -x_1(v) - (x_2^{\frac{1}{2}}(v) + {}^{R}D^{\gamma_1}x_2)(x_2^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v)) \\ x_2(v) - (x_2^{\frac{1}{3}}(v) + {}^{R}D^{\gamma_1}x_1)(x_1^{\frac{1}{2}}(v) + I^{\gamma_2}x_2(v)) \end{bmatrix} \leq -\|x(v)\|^{\frac{4}{3}}$

Therefore, the system is globally exponentially stable.

Example 1.2. Consider the following compost equation fractional which is defined as follows:

$$^{R}D^{\alpha_1}{}^{R}D^{\alpha_2}\big(X(t)\big) = \eta sgn(X(t)) \qquad 0 < \alpha_1 < \alpha_2 < 1 \,, \ \eta > 0$$

Using Lyapunov function

$$V(t) = \frac{1}{2}x^2(t)$$

$$\dot{V} = \frac{1}{2}\big(2x(t)\dot{x}(t)\big)$$

$$\dot{V}(t) = x(t)\dot{x}(t)$$

$$^{R}D^{\alpha_1}{}^{R}D^{\alpha_2}\big(x(t)\big) = \begin{cases} > 0 \ if \ \dot{x}(t) > 0 \\ < 0 \ if \ \dot{x}(t) < 0 \end{cases}$$

$$^{R}D^{\alpha_1}{}^{R}D^{\alpha_2}\big(x(t)\big) = -\eta sgn\,(x(t)) \qquad , \qquad 0 < \alpha_1 < 1 \quad , \quad 0 < \alpha_2 < 1$$

$$^{R}D^{\alpha_1+\alpha_2}x(t) - \big[{}^{R}D_t^{\alpha_2-1}x(t)\big]_{t=0}\frac{t^{-\alpha_1-1}}{\Gamma(1-\alpha_1-1)} = -\eta sgn(x(t))$$

$$^{R}I^{\alpha_1+\alpha_2}\ {}^{R}D^{\alpha_1+\alpha_2}x(t) - {}^{R}I^{\alpha_1+\alpha_2}\left({}^{R}D_t^{\alpha_2-1}x(t)\big|_{t=0}\right)\frac{t^{-\alpha_1-1}}{\Gamma(-\alpha_1)} = -\eta\,{}^{R}I^{\alpha_1+\alpha_2}(sgn\ x(t))$$

$$X(t) - X(0) = \begin{cases} {}^{R}I^{\alpha_1+\alpha_2}\left({}^{R}D_t^{\alpha_2-1}x(t)\big|_{t=0}\right)\dfrac{t^{-\alpha_1-1}}{\Gamma(-\alpha_1)} - \eta\left(\dfrac{1-\tau}{M(\tau)} + \dfrac{\tau\,t}{M(\tau)}\right), X(t) > 0 \\[4mm] {}^{R}I^{\alpha_1+\alpha_2}\left({}^{R}D_t^{\alpha_2-1}x(t)\big|_{t=0}\right)\dfrac{t^{-\alpha_1-1}}{\Gamma(-\alpha_1)} + \eta\left(\dfrac{1-\tau}{M(\tau)} + \dfrac{\tau\,t}{M(\tau)}\right), X(t) < 0 \end{cases}$$

$$\dot{X}(t) = \begin{cases} D^{R}I^{\alpha_1+\alpha_2}{}^{R}D_t^{\alpha_2-1}x(t)\big|_{t=0}\dfrac{t^{-\alpha_1-1}}{\Gamma(-\alpha_1)} - \eta\dfrac{\tau}{M(\tau)} \ \ if \ sgn(x(t)) > 0 \\[4mm] D^{R}I^{\alpha_1+\alpha_2}{}^{R}D_t^{\alpha_2-1}x(t)\big|_{t=0}\dfrac{t^{-\alpha_1-1}}{\Gamma(-\alpha_1)} + \eta\dfrac{\tau}{M(\tau)} \ \ if \ sgn(x(t)) < 0 \end{cases}$$

If $\left. D^R I^{\alpha_1+\alpha_2 \, R} D_t^{\alpha_2-1} x(t) \right|_{t=0} \dfrac{t^{-\alpha_1-1}}{\Gamma(-\alpha_1)} < \eta \dfrac{\tau}{M(\tau)}$

and $\eta \dfrac{\tau}{M(\tau)} > 0$

Then

$$\dot{X}(t) = \begin{cases} < 0 & if \ sgn(x(t)) > 0 \\ > 0 & if \ sgn(x(t)) < 0 \end{cases}$$

Thus

$$sgn \, \dot{X}(t) = -sgn \, x(t)$$

We have that $sgn\left(\dot{v}(t)\right) = sgnX(t)sgn\left(\dot{X}(t)\right) = -sgnX(t)sgn\dot{X}(t) < 0$

So, if $\dot{v}(t) < 0$ then ${}^R D^{\alpha_1 \, R} D^{\alpha_2} v(t) < 0$

Then by lemma (1.1), $x = 0$ is globally exponentially stable.

Conclusions:

This paper aims to present the Lyapunov functions in the stability and stabilization of different types of fractional deferential and integro-differential fractional related to some nonlinear functions of fractional and integral components including different types of fractional derivatives all the results explained by fractional analytic made different treatments than ordinary analytic. The systems were presented as a new formulation of stabilization field such as compositions Riemann-Liouville.

Some examples were presented to explain the new results of this paper.

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