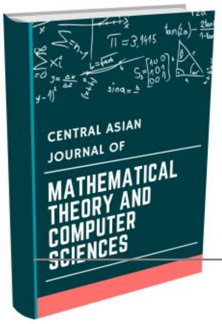

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UNVEILING THE POWER OF SOBOLEV SPACES FOR ENGINEERING APPLICATIONS IN LIPSCHITZ-BOUNDED DOMAINS

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Abstract

A family of universal extension operators for Sobolev spaces of differential forms on Lipschitz-bounded domains is the major goal of this paper. We want to show that Sobolev spaces of differential forms are essential to the analysis of bounded Lipschitz domains. The goal is to connect mathematics theory to engineering and science applications in complicated geometric contexts. Sobolev spaces, a staple of mathematical analysis, are widely used to explore partial differential equations and provide attractive Euclidean solutions. These conventional spaces struggle with Lipschitz-limited domains due to their complicated geometric properties. Universal extension operators provide a systematic and robust way to apply Sobolev spaces to more complex scenarios. Within the scope of this research, we investigate the theory of Sobolev spaces for differential forms as well as their application to Lipschitz-constrained domains. The construction approach maintains essential aspects of mathematics by building atop smooth forms, L2-spaces, and Sobolev spaces. Sobolev spaces have wide-ranging applications across engineering and science. In structural mechanics, they model stress distribution in irregular components, optimizing material use in constructions like reinforced concrete beams. For image processing, Sobolev spaces enable precise denoising, vital in medical imaging and satellite analysis. Electrical engineers employ Sobolev-based methods for electromagnetic field analysis in designing antennas and circuits. In aerospace, Sobolev spaces simulate airflow around complex surfaces, improving aerodynamics. In material science, they aid in understanding nanoscale behaviors, advancing quantum properties and nanomaterial applications. These case studies highlight Sobolev spaces' adaptability, providing a mathematical foundation to tackle intricate challenges across scientific and engineering disciplines.

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I. INTRODUCTION

The interplay between mathematics and the practical demands of engineering and science is an enduring and fruitful partnership. In the intricate landscapes of engineering challenges and scientific inquiry, mathematical theory provides a powerful set of tools to address complex real-world problems. This paper embarks on a journey to explore the nexus of mathematical rigor and practical applications by focusing on the construction of universal extension operators for Sobolev spaces of differential forms within Lipschitz-bounded domains [1-4].

Sobolev spaces, well-established in the realm of mathematical analysis, have long been instrumental in providing elegant solutions to partial differential equations within the familiar context of Euclidean spaces. However, as we venture into the domain of Lipschitz-bounded regions, we are confronted with complex geometric properties and irregular boundaries that challenge the applicability of conventional Sobolev spaces. It is precisely in this challenging terrain that we seek to introduce universal extension operators as a systematic and robust solution to extend the capabilities of Sobolev spaces [5-7].

The use of Sobolev spaces with fractional order derivatives and integrals can broaden the applicability of the mathematical tools introduced in the future [8-10]. This may open new possibilities for solving differential equations or analyzing complex geometric domains with non-integer order derivatives, which can model various physical processes more accurately [11-20]. Fractional calculus has applications in physics, engineering, biology, and many other fields. By incorporating this aspect, the paper underlines the interdisciplinary significance of Sobolev spaces and universal extension operators in addressing a wider range of real-world problems [21-25]. Our primary goal is twofold: first, to showcase that Sobolev spaces of differential forms are not merely theoretical abstractions but fundamental tools for the analysis of bounded Lipschitz domains, and second, to establish a bridge between the rigor of mathematical theory and the practical demands of engineering and science in intricate geometric contexts [26-33].

Sobolev spaces are crucial in mathematical analysis, especially for handling functions with irregularities or in complex geometric spaces, essential for understanding partial differential equations. In Lipschitz-bounded domains, where traditional solutions fall short due to intricate geometry, Sobolev spaces become pivotal. Universal extension operators offer a systematic approach to apply Sobolev spaces in these domains, unlocking solutions that were previously inaccessible. This paper closes the gap between theoretical math and practical engineering/science by exploring Sobolev spaces for differential forms in Lipschitz-constrained domains. It not only enriches mathematical theory but also showcases their vital role in solving real-world problems in civil and electrical engineering. These spaces act as a mathematical bridge, linking rigorous theory to practical applications to tackle intricate challenges in engineering and science.

Constructing universal extension operators in Lipschitz-bounded domains blends principles from differential geometry, functional, and harmonic analysis. Understanding the domain's complex geometry leads to developing lifting operations mappings extending smooth forms across the entire domain while preserving essential traits. These underpin extension operators, precisely crafted to extend functions within the domain while maintaining Sobolev regularity.

Harmonic analysis techniques, like integral operators, ensure extended functions comply with domain constraints. Rigorous proofs establish these operators' universality, extending functions from diverse subdomains while preserving crucial properties. Validation via mathematical analysis and simulations confirms their effectiveness in solving differential equations in Lipschitz domains.

Ultimately, the aim is to generalize these operators for broader engineering and scientific applications, bridging theoretical mathematics with practical challenges in Lipschitz-constrained domains.

This study explores the idea of Sobolev spaces for differential forms and their applicability to domains with Lipschitz constraints to accomplish these goals. Our proposed building method is grounded in fundamental mathematical concepts including smooth forms, L2-spaces, and Sobolev spaces. By introducing extension operators, we equip engineers and scientists with powerful new resources for tackling difficult issues in fields as diverse as structural mechanics in civil engineering and electromagnetic field analysis in electrical engineering [34, 35]. It is impossible to exaggerate the significance of Sobolev spaces of differential forms in today's engineering and scientific world. These domains facilitate the systematic modeling, analysis, and optimization of complex systems and processes, acting as a bridge between mathematical rigor and technological progress. In this paper, we show how Sobolev spaces of differential forms play a crucial role in meeting the ever-evolving challenges of engineering and scientific investigation by demonstrating how the application of universal extension operators helps academics and practitioners navigate the complexities of Lipschitz-bounded domains [36-40].

Empirical evidence showcasing the efficacy of extension operators on irregular boundaries spans various fields. In structural mechanics, experiments or simulations comparing stress models demonstrate how these operators enhance predictions in irregular components, optimizing material usage. In medical imaging, datasets with irregular noise patterns validate extension operators' superiority in preserving image details while reducing noise. Similarly, in electromagnetic field analysis, simulations for irregular antenna designs reveal enhanced accuracy and performance. Computational fluid dynamics studies in aerospace exhibit improved aerodynamic predictions around irregular surfaces. Moreover, empirical setups exploring nanoscale behaviors in irregular structures showcase extension operators' role in accurately describing quantum properties. These examples collectively affirm the operators' superiority in handling irregular boundaries, validating their crucial role across diverse scientific and engineering domains.

As we set out on our adventure, it's worth remembering that mathematics' appeal comes from more than just its abstract beauty; it also has the power to influence and progress the material worlds of engineering and science. Through this exploration, we seek to strengthen the bond between theory and practice, offering a path for engineers and scientists to address complex real-world challenges with mathematical precision [41-50].

II. UNIVERSAL EXTENSION OF DIFFERENTIAL FORMS

Our focus is on the development of universal extension operators for Sobolev spaces with differential forms. In this discussion, we will begin by providing a concise overview of the fundamental components of Stein's method for constructing the universal extension operator for standard Sobolev spaces $H^k(\Omega)(k \in \mathbb{N}_0)$ [47]). Subsequently, we will demonstrate the extension for the scenario involving a Lipschitz epigraph, incorporating many crucial elements. Finally, we will proceed to extend this generalization to bound Lipschitz domains through the utilization of the partition of unity technique.

A. Some Applied Theories and Technical Lemmas:

In the subsequent Lemma, we present a regularized distance that exhibits improved smoothness as an alternative to $\delta(x)$.

Lemma 1 [47]: there exists a regularized distance function $\Delta(x) = \Delta(,\overline{\Omega})$ such that for $x \in \overline{\Omega}^c$, For a closed domain $\overline{\Omega} \in \mathbb{R}^d$,

i $c\delta(x) \leq \Delta(x) \leq C\delta(x)$

ii $\Delta(x)$ is C^∞- smooth in $\overline{\Omega}^c$ and $\left|\frac{\partial^\alpha}{\partial x^\alpha}\Delta(x)\right| \leq C_\alpha(\delta(x))^{1-|\alpha|}$,

where c > 0 and C >0 are constants independent of $\overline{\Omega}$ and $C_\alpha > 0$ depends on the multi-index α^2.

The following two technical lemmas are key tools to construct universal extension operators.

Lemma 2. [47]. The weighting function

$$\psi(\lambda) = \frac{e}{\pi\lambda}\Im\left(\exp\left(\frac{1}{2}\sqrt{2}(-`1 + i)(\lambda - 1)^{\frac{1}{4}}\right)\right), \qquad (2)$$

$$\psi(\lambda) = O(\lambda^{-n}) \quad \text{as } \lambda \to \infty, \forall n \in \mathbb{N},$$

and all its higher moments vanish

$$\int_1^\infty \lambda^k \psi(\lambda) d\lambda = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k \in \mathbb{N}\,. \end{cases} \tag{3}$$

Now we consider the special case that Ω is a Lipschitz epigraph with its boundary defined by a Lipschitz function : $\mathbb{R}^{d-1} \mapsto \mathbb{R}$. We split position vectors according to $x = (\hat{x}, y) \in \mathbb{R}^d$, where $\hat{x} \in \mathbb{R}^{d-1}$, and $y \in \mathbb{R}$.

Lemma 3[47] For a Lipschitz epigraph Ω , let $\Delta(x)$ be the regularized distance given in Lemma 1. Then there exists a constant $C_\delta = C_\delta(\phi) > 0$ such that for $x = (\hat{x}, y) \in \overline{\Omega}^C$.

$$C_\delta \Delta(x) \geq \phi(\hat{x}) - y. \tag{4}$$

We define a scaled smoothed distance $\delta^*(x) = 2C_\delta\Delta(x)$ with smoothness inherited from $\Delta(x)$. from (4) it is immediate to see that

$$\delta^*(x) \geq 2(\phi(\hat{x}) - y) \tag{5}$$

III. RESULTS AND DISCUSSION

A. Extension formula for Lipchitz Domain

The classical Stein extension formula [47] for compactly supported smooth functions f on a Lipchitz domain $\overline{\Omega}$ reads

$$\mathfrak{H}(f)(x) = \int_1^\infty f\big(\hat{x}, y + \lambda\delta^*(x)\psi(\lambda)\big) d\lambda \tag{6}$$

To generalize this formula, let us first define a parametrized reflection mapping for

$$\begin{aligned} &x = (\hat{x}, y) \in \overline{\Omega}^C \in \mathbb{R}^d, \\ &\mathfrak{R}_\lambda(x) = \big(\hat{x}, y + \lambda\, \delta^*(x)\big) = x + \lambda\, \delta^*(x) e_d. \end{aligned} \tag{7}$$

Note that for points $x = (\hat{x}, y) \in \overline{\Omega}$ we have, using the fact that $\delta^*(x) = 0$,

$$\mathfrak{R}_\lambda(x) = (\hat{x}, y + 0) = x.$$

In other words, $\mathfrak{R}_\lambda$ reduces to the identity operator in $\overline{\Omega}$. However, for $x = (\hat{x}, y) \in \overline{\Omega}^C$ with $y < \phi(\hat{x})$, due to (5) and the fact that) $\lambda \geq 1$, we see that

$$y + \lambda\delta^*(x) \geq y + 2(\phi(\hat{x}) - y) \geq \phi(\hat{x}) + (\phi(\hat{x}) - y) > \phi(\hat{x})\,.$$

Thus, the parametrized reflection mapping $\mathfrak{R}_\lambda$ always maps $x \in \overline{\Omega}^C$ into Ω for any $\lambda \in [1, \infty)$. It is straightforward to calculate the Jacobian of the parametrized reflection mapping

$$D\mathfrak{R}_\lambda(x) = \begin{pmatrix} Id_{d-1} & 0 \\ \lambda \text{grad}_{\hat{x}} \delta^*(x)^T & 1 + \lambda \frac{\partial \delta^*(x)}{\partial x_d} \end{pmatrix}, \tag{8}$$

where $\text{grad}_{\hat{x}}\, \delta^*(x) = (\frac{\partial \delta^*(x)}{\partial x_1}, \ldots\ldots \frac{\partial \delta^*(x)}{\partial x_{d-1}})^T$ and 0 represents a column vector with (d-1) zeros.

The function f in (6) can be regarded as a vector proxy of a compactly supported 0-form ω on $\overline{\Omega}$. From this perspective, $x \mapsto f\,(\hat{x}, y + \lambda\delta^*(x))$ turns out to be the vector proxy of the pullback $\mathcal{R}_\lambda^* f$.This immediately suggests the following generalization of (6) to a universal extension operator for smooth compactly supported l-forms on $\overline{\Omega}$

$$(\mathfrak{H}_l\omega)(x) := \begin{cases} \omega(x), & x \in \overline{\Omega}, \\ \int_1^{\infty} (\mathfrak{R}_\lambda^* \omega)(x)\psi(\lambda)d\lambda, & x \in \overline{\Omega}^c. \end{cases} \quad (9)$$

We fix an increasing l-permutation $I = (i_1, \dots, i_l)$ with $i_1 < \cdots < i_l \leq d$. For a compactly supported differential l -form $\omega \in \mathcal{DF}^{l,\infty}(\overline{\Omega})$ we have

$$(\mathfrak{H}_l\omega)_I(x) := (\mathfrak{H}_l\omega)(x)\big(e_{i_1}, \dots, e_{i_l}\big) = \int_1^{\infty} (\mathfrak{R}_\lambda^*\omega)(x)\big(e_{i_1}, \dots, e_{i_l}\big)\,\psi(\lambda)d\lambda$$
$$= \int_1^{\infty} \Big(\omega\big(\mathfrak{R}_\lambda(x)\big)\Big)\,(D\mathfrak{R}_\lambda(x)e_{i_1}, \dots, D\mathfrak{R}_\lambda(x)e_{i_l})\,\psi(\lambda)d\lambda.$$

From (8) we infer

$$(D\mathfrak{R}_\lambda(x))e_{i_k} = e_{i_k} + \lambda\frac{\partial\delta^*(x)}{\partial x_{i_k}}e_d \text{ for } 1 \leq k \leq l, \quad (10)$$

We obtained

$$(\mathfrak{H}_l\omega)_I(x) = \mathfrak{K} + \sum_{k=1}^{l} (-1)^{l-k}\mathfrak{J}_{i_k}, \quad (11)$$

Where we have used the abbreviations

$$\mathfrak{K} := \int_1^{\infty} \Big(\omega_I\big(\mathfrak{R}_\lambda(x)\big)\Big)\,\psi(\lambda)d\lambda,$$

$$\mathfrak{J}_i := \frac{\partial\delta^*(x)}{\partial x_i}\int_1^{\infty} \Big(\omega_{I_i\cup\{d\}}\big(\mathfrak{R}_\lambda(x)\big)\Big)\lambda\psi(\lambda)d\lambda, \quad i = 1,2, \dots, d,$$

and by $\tilde{I}_{i_k} \cup \{d\}$ we designate the increasingl-permutation $1 \leq i_1 < \cdots < \tilde{i}_k < \cdots i_l < d$ with i_k dropped and d included. For $i_l = d$, we have a simpler representation, viz,

$$(\mathfrak{H}_l\omega)_I(x) = \mathfrak{K} + \mathfrak{J}_d. \quad (12)$$

For $d\omega$, using the commuting diagram property of exterior derivative and the parameterized reflection mapping $\mathfrak{R}_\lambda$ used in $\mathfrak{H}_l$, we derive

$$d(\mathfrak{H}_l\omega)(x) = \int_1^{\infty} d(\mathfrak{R}_\lambda^*\omega)(x)\psi(\lambda)d\lambda = \int_1^{\infty} \mathfrak{R}_\lambda^*\,(d\omega)(x)\psi(\lambda)d\lambda. \quad (13)$$

This implies

$$d \circ \mathfrak{H}_l = \mathfrak{H}_{l+1} \circ d. \quad (14)$$

Before we proceed, we must verify that $\delta_l\omega$ provides well-defined differential *L*–forms. See Fig. 1 to illustrate.

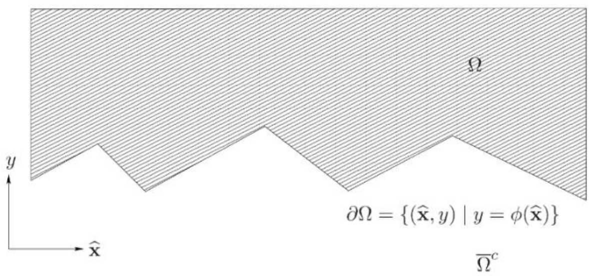

Fig. 1 Sketch of a Lipschitz epigraph.

Lemma 4 For a Lipschitz epigraph Ω, the extension formula in (9) is well- defined in the sense that for compactly supported $\omega \in \mathcal{DF}^{1,\infty}(\overline{\Omega})$,

$$\delta_l\omega = \omega \text{ in } \overline{\Omega} \text{ and } \delta_l\omega \in \mathcal{DF}^{1,\infty}(\mathbb{R}^d) \qquad (15)$$

Proof. The bounded support and smoothness of ω guarantee that $\delta_l\omega$ is well - defined everywhere in$\mathbb{R}^d$. In particular, $\delta_l\omega = \omega$ in $\overline{\Omega}$ due to the fact that the reflection mapping $\mathcal{R}_\lambda$ reduces to the identity operator.

The smoothness of δ^* and ω along with the compact support of ω ensures that $\delta_l\omega$ belongs to $\mathcal{DF}^{1,\infty}$ ($\Omega \cup \Omega^c$). We have to prove that all partial derivatives $\frac{\partial^\alpha (\delta_l\omega)_I}{\partial x^\alpha}$ for any multi-index αand any component index I are continuous across ∂Ω.

Now, we establish continuity of both $\frac{\partial^2 \mathfrak{R}}{\partial x_j^2}$ and $\frac{\partial^2 \mathfrak{I}_i}{\partial x_j^2}$ across $\partial\Omega$: Let $x = (\hat{x}, y) \in \overline{\Omega}^c$ tend to some point $x^0 = (\hat{x}^0, y^0)$ on the boundary $\partial\Omega$, that is, $y^0 = \phi(\hat{x}^0)$. Then by Lemma 3, $\delta^*(x) \to 0$ and the derivatives $\frac{\partial\delta^*(x)}{\partial x_j}$, $1 \le j \le d$, are bounded uniformly as $x \to x^0$. By Lemma 2.2 the first three terms on the right hand side of (15) converge to $\frac{\partial^2 \omega_I}{\partial x_j^2}(x^0)$, 0 and 0, respectively.

As for the last term in (15), the difficulty involving the unboundedness of the higher order derivatives of δ^* can be circumvented by using the Taylor expansion of ω_I about $(\hat{x}, y + \delta^*)$:

$$\frac{\partial \omega_I(\mathcal{R}_\lambda(x))}{\partial x_d} = \frac{\partial \omega_I(\hat{x}, y+\delta^*)}{\partial x_d} + (\lambda - 1)\delta^*(x)\frac{\partial^2 \omega_I(\hat{x}, y+\delta^*)}{\partial x_d^2} + r(\lambda, x), \qquad (16)$$

with a remainder term $r(\lambda, x)$ that satisfies

$$|r(\lambda, x)| \le C[(\lambda - 1)\delta^*(x)]^2 \forall x \in \Omega^c, \lambda > 1. \qquad (17)$$

Returns to Lemma 1 we conclude

$$\left| r(\lambda, x)\frac{\partial^2 \delta^*(x)}{\partial x_j^2} \right| \le C(\lambda - 1)^2 \delta^*(x) \forall x \in \Omega^c, \lambda > 1. \qquad (18)$$

Hence, substituting the identity (17) into (15) gives two more vanishing integrals plus a remainder

$$\left|\int_1^\infty \mathrm{r}(\lambda,\mathrm{x})\lambda\frac{\partial^2\delta^*(\mathrm{x})}{\partial \mathrm{x}_j^2}\psi(\lambda)\mathrm{d}\lambda\right| \leq \mathrm{C}\delta^*(\mathrm{x})\int_1^\infty(\lambda-1)^2\lambda|\psi(\lambda)|\mathrm{d}\lambda \to 0\,, \tag{19}$$

since the last integral is uniformly bounded by (1) and $\delta^*(\mathrm{x}) \to 0$ as $\mathrm{x} \to \mathrm{x}_0$.

See Figure 2 to illustrate.

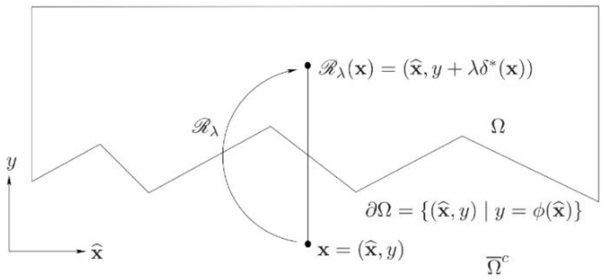

Fig. 2 Parametrized mapping for Lipschitz epigraph.

B. Continuity of extension operators

Theorem 1. Let Ω be a Lipschitz epigraph in $\mathbb{R}^d, k \in \mathbb{N}_0$ and $0 \leq l \leq d$. Then the extension operator (14) satisfies

$$\|\mathcal{E}_l\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \leq C \parallel \boldsymbol{\omega} \parallel_{\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)},$$

$\forall$ compactly supported $\boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\bar{\Omega})$, with a constant $C = C(\Omega, d, k, l) > 0$. Thus, $\mathcal{E}_l$ can be extended to a continuous extension operator

$$\mathcal{E}_l\colon H^k(d,\Omega,\Lambda^l) \mapsto H^k(d,\mathbb{R}^d,\Lambda^l). \tag{20}$$

Proof. The second assertion relies on density argument, because compactly supported differential forms in $\mathcal{DF}^{l,\infty}(\bar{\Omega}) \cap H^k(d,\Omega,\Lambda^l)$ form a dense subset of $H^k(d,\Omega,\Lambda^l)$.

Then from (16) (the argument for (17) is the same)

$$|(\mathcal{E}_l\boldsymbol{\omega})_I(\hat{\mathbf{x}}^0,y)| \leq C\left(\int_1^\infty |\boldsymbol{\omega}_I(\mathcal{R}_\lambda(\mathbf{x}))|\frac{1}{\lambda^2}\,\mathrm{d}\lambda + \sum_{k=1}^{l}\int_1^\infty \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\mathcal{R}_\lambda(\mathbf{x}))\right|\frac{1}{\lambda^2}\,\mathrm{d}\lambda\right), \text{ for } y < 0, \tag{21}$$

where we have used the facts that $|\psi| \leq C_1/\lambda^2$ for the first term and $|\psi| \leq C_2/\lambda^3$ for the summation term in the right hand side, and that $\left|\frac{\partial\delta^*(\mathbf{x})}{x_{i_k}}\right| \leq C$ from Lemma 1.

For fixed $\mathrm{x} = (\hat{\mathrm{x}}, y)$ with $y < 0$, we have $\delta^*(\mathrm{x}) \geq 2|y|$ for $\phi(\hat{\mathrm{x}}^0) = 0$ by (2.5). On the other hand, by Lemma 1, we see $\delta^*(\mathrm{x}) = C\Delta(\mathrm{x}) \leq C \cdot \tilde{C}\delta(\mathrm{x}) \leq C \cdot \tilde{C}|y|$ since $\phi(\hat{\mathrm{x}}^0) - y$ is not less than the distance $\delta(\hat{\mathrm{x}}, y)$ of $(\hat{\mathrm{x}}, y)$ from $\bar{\Omega}$. By performing the change of variables $s = y + \lambda\delta^*(\mathrm{x})$, then we have $\mathrm{d}s = \delta^*(\mathrm{x})\mathrm{d}\lambda$ and

$$\begin{aligned}&\int_1^\infty \left|\left(\omega_I(\mathcal{R}_\lambda(\mathrm{x}))\right)\right| \frac{1}{\lambda^2}\, \mathrm{d}\lambda \\ &\leq C|y| \int_{|y|}^\infty \left|\left(\omega_I(\hat{\mathrm{x}}^0, s)\right)\right| \frac{1}{s^2}\, \mathrm{d}s.\end{aligned} \tag{22}$$

Note that $(s-y) \geq s$ for $y<0$ and $s>0$. Likewise, it still holds when we replace I by $\check{I}_{i_k} \cup \{d\}$ for $k=1,\ldots,l$.

Recall the Hardy inequality [47,pp. 272]:, for any $f \geq 0, p \geq 1$ and $r>0$,

$$\begin{aligned}&\left(\int_0^\infty \left(\int_x^\infty f(y)\mathrm{d}y\right)^p x^{r-1}\, \mathrm{d}x\right)^{1/p} \\ &\leq \frac{p}{r}\left(\int_0^\infty (yf(y))^p y^{r-1}\, \mathrm{d}y\right)^{1/p}.\end{aligned} \tag{23}$$

We can apply (23) for the case that $r=3$ and $p=2$ to obtain

$$\begin{aligned}&\left(\int_{-\infty}^0 |(\mathcal{E}\boldsymbol{\omega})_I(\hat{\mathbf{x}}^0, y)|^2\, \mathrm{d}y\right)^{1/2} \\ &\leq C\left(\int_{-\infty}^0 \left(\int_{|y|}^\infty \left(|\boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)|\right.\right.\right. \\ &\left.\left.\left. + \sum_{k=1}^l \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\hat{\mathbf{x}}^0, s)\right|\right)\frac{1}{s^2}\, \mathrm{d}s\right)^2 |y|^2\, \mathrm{d}y\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{-\infty}^0 \left(\left(|\boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, |y|)|\right.\right.\right. \\ &\left.\left.\left. + \sum_{k=1}^l \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\hat{\mathbf{x}}^0, |y|)\right|\right)\frac{|y|}{|y|^2}\right)^2 |y|^2\, \mathrm{d}y\right)^{\frac{1}{2}} \\ &\leq C\left(\int_0^\infty \left(|\boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, y)|^2\right.\right. \\ &\left.\left. + \sum_{k=1}^l \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\hat{\mathbf{x}}^0, y)\right|^2\right)\mathrm{d}y\right)^{\frac{1}{2}},\end{aligned} \tag{24}$$

where we use the Hardy inequality for the second inequality, the Cauchy-Schwarz inequality. and the change of variable $y \to -y$ in the last.

Furthermore, the assumption $\phi(\hat{\mathbf{x}}^0)=0$ can be dropped by using an appropriate translation in y, which yields

$$\left(\int_{-\infty}^{\infty} |(\mathcal{E}_l\omega)_I(\hat{x}^0,y)|^2 \,\mathrm{d}y\right)^{1/2} \leq \left(\left(\int_{-\infty}^{\phi(\hat{x}^0)} + \int_{\phi(\hat{x}^0)}^{\infty}\right)|(\mathcal{E}\omega)_I(\hat{x}^0,y)|^2 \,\mathrm{d}y\right)^{1/2} \leq C\left(\int_{\phi(\hat{x}^0)}^{\infty}\left(|\omega_I(\hat{x}^0,y)|^2 + \sum_{k=1}^{l}\left|\omega_{\check{I}_{i_k}\cup\{d\}}(\hat{x}^0,y)\right|^2\right)\mathrm{d}y\right)^{\frac{1}{2}} \tag{25}$$

Taking the square on both sides and integrating over all $\hat{\mathbf{x}} \in \mathbb{R}^{d-1}$ yields

$$\|(\mathcal{E}_l\boldsymbol{\omega})_I\|^2_{L^2(\mathbb{R}^d;\Lambda^l)} \leq C\left(\|\boldsymbol{\omega}_I\|^2_{L^2(\Omega;\Lambda^l)} + \sum_{k=1}^{l}\left\|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}\right\|^2_{L^2(\Omega;\Lambda^l)}\right). \tag{26}$$

Summing over all indices I we derive

$$\|\mathcal{E}_l\boldsymbol{\omega}\|^2_{L^2(\mathbb{R}^d;\Lambda^l)} \leq C \parallel \boldsymbol{\omega} \parallel^2_{L^2(\Omega;\Lambda^l)}.$$

In exactly the same way, we can show that, in view of the commuting diagram property (14),

$$\|d(\mathcal{E}_l\omega)\|_{L^2(\mathbb{R}^d;\Lambda^{l+1})} = \|\mathcal{E}_{l+1}(d\omega)\|_{L^2(\mathbb{R}^d;\Lambda^{l+1})} \leq C \parallel d\omega \parallel_{L^2(\Omega;\Lambda^{l+1})},$$

which completes the proof in the case $k = 0$.

The proof for $k > 0$ is again done for one representative special case. Let us take $k = 2$ with $\frac{\partial^2((\mathcal{E}_l\boldsymbol{\omega})_I)}{\partial x_j^2}$ as our specimen. Using $\psi(\lambda) \leq C_1/\lambda^2, C_2/\lambda^3, C_3/\lambda^4$, respectively, the terms in (15) can be bounded as follows:

$$\left|\frac{\partial^2(\mathcal{E}_l\omega)_I(\hat{x}^0,y)}{\partial x_j^2}\right| \leq C\int_1^{\infty}\left(\left|\left(\frac{\partial^2\omega_I(\hat{x}^0,y)}{\partial x_j^2}\right)\right| + \left|\left(\frac{\partial^2\omega_I(\hat{x}^0,y)}{\partial x_j x_d}\right)\right| + \left|\left(\frac{\partial^2\omega_I(\hat{x}^0,y)}{\partial x_d^2}\right)\right|\right)\frac{1}{\lambda^2}\,\mathrm{d}\lambda + \left|\int_1^{\infty}\left(\frac{\partial\omega_I\left(\mathcal{R}_\lambda(\hat{x}^0,y)\right)}{\partial x_d}\right)\lambda\frac{\partial^2\delta^*(\hat{x}^0,y)}{\partial x_j^2}\psi(\lambda)\mathrm{d}\lambda\right|. \tag{27}$$

Theorem 2. Let Ω be a Lipschitz epigraph in $\mathbb{R}^d$, $k \in \mathbb{N}$ and $0 \leq l \leq d$. Then the extension operator (16) satisfies

$$\|\delta_l \omega\|_{H^k(d,\mathbb{R}^d,\Lambda^l)} \leq C\|\omega\|_{H^k(d,\mathbb{R}^d,\Lambda^l)}$$

with a constant $C = C(\Omega, d, k, l) > 0$. Thus, $\delta_l$can be extended to a continuous extension then

$$\begin{aligned}&\frac{\partial \omega_I}{\partial x_d}\left(\mathcal{R}_\lambda(\hat{x}^0, y)\right)\\&= \frac{\partial \omega_I}{\partial x_d}\left(\hat{x}^0, y + \delta^*(\hat{x}^0, y)\right)\\&+ \int_{y+\delta^*(\hat{x}^0,y)}^{y+\lambda\delta^*(\hat{x}^0,y)} \frac{\partial^2 \omega_I}{\partial x_d^2}(\hat{x}^0, s)\mathrm{d}s.\end{aligned} \tag{28}$$

Substituting this in (26), we know that the integral term involving $\frac{\partial \omega_I}{\partial x_d}\left(\hat{\mathbf{x}}^0, y + \delta^*(\hat{\mathbf{x}}^0, y)\right)$ vanishes due to Lemma 2. Hence, it suffices to show the following bound (Note that $\left|\frac{\partial^2 \delta^*}{x_j^2}(\hat{\mathbf{x}}^0, y)\right| \leq C|\delta(\hat{\mathbf{x}}^0, y)|^{-1} \leq C|y|^{-1}$. We assume $\phi(\hat{\mathbf{x}}^0) = 0$ without loss of generality)

$$\begin{aligned}&|y|^{-1} \int_1^\infty \left\{\int_{y+\delta^*(\hat{\mathbf{x}}^0,y)}^{y+\lambda\delta^*(\hat{\mathbf{x}}^0,y)} \left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right| \mathrm{d}s\right\} \frac{1}{\lambda^3}\, \mathrm{d}\lambda\\&= |y|^{-1} \int_{y+\delta^*(\hat{\mathbf{x}}^0,y)}^{\infty} \left\{\int_{\frac{s-y}{\delta^*(\hat{\mathbf{x}}^0,y)}}^{\infty} \left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right| \frac{1}{\lambda^3}\, \mathrm{d}\lambda\right\} \mathrm{d}s\\&\leq |y|^{-1}\left(\delta^*(\hat{\mathbf{x}}^0, y)\right)^2\\&\qquad \int_{y+\delta^*(\hat{\mathbf{x}}^0,y)}^\infty \left\{\left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right|\right\} \frac{1}{(s-y)^2}\mathrm{d}s\\&\qquad \leq C|y| \int_{|y|}^\infty \left\{\left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right|\right\} \frac{1}{s^2}\, \mathrm{d}s\end{aligned} \tag{29}$$

where we have interchanged the order of integration for the first equality, and used that $\delta^*(\hat{\mathbf{x}}^0, y) \leq C|y|, \delta^*(\hat{\mathbf{x}}^0, y) \geq 2|y|$ and $s - y \geq s$ when $y < 0$ for the second inequality. Thus we can appeal to the Hardy inequality once again for (24) and integrate over all $\hat{\mathbf{x}} \in \mathbb{R}^{d-1}$ to obtain

$$\begin{aligned}&\left\|\frac{\partial^2 (\mathcal{E}_l \boldsymbol{\omega})_I}{\partial x_j^2}\right\|_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^l)}^2\\&\qquad \leq C\left(\left\|\frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_j^2}\right\|_{\boldsymbol{L}^2(\Omega;\Lambda^l)}^2\right.\\&\qquad + \left\|\frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_j x_d}\right\|_{\boldsymbol{L}^2(\Omega;\Lambda^l)}^2\\&\qquad \left.+ \left\|\frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_d^2}\right\|_{\boldsymbol{L}^2(\Omega;\Lambda^l)}^2\right).\end{aligned} \tag{30}$$

Analogously by the commuting diagram property, we have

$$\left\|\frac{\partial^2 d(\mathcal{E}\omega)_I}{\partial x_j^2}\right\|^2_{L^2(\mathbb{R}^d;\Lambda^{l+1})} \leq C\left(\left\|\frac{\partial^2 d\omega_I}{\partial x_j^2}\right\|^2_{L^2(\Omega;\Lambda^{l+1})} + \left\|\frac{\partial^2 d\omega_I}{\partial x_j\,\partial x_d}\right\|^2_{L^2(\Omega;\Lambda^{l+1})} + \left\|\frac{\partial^2 d\omega_I}{\partial x_d^2}\right\|^2_{L^2(\Omega;\Lambda^{l+1})}\right). \quad (31)$$

Thus we have proved the assertion for the case$k = 2$.

C. Regular decompositions

This section establishes regular decompositions of differential form Sobolev spaces using the universal extension result. Thus, a well-known lifting lemma applies to differential form Sobolev spaces. We assume $d \geq 3$ and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and only discuss $d \geq 3$ in this section.

Regular decomposition rely on the existence of regular potentials in $\mathbb{R}^d$.

Lemma 5 (Existence of regular potentials in $\mathbb{R}^d$). For $1 \leq l \leq d, l \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ there is a continuous lifting mapping

$$\mathcal{L}: H(d0, \mathbb{R}^d, \Lambda^l) \cap H^k(\mathbb{R}^d, \Lambda^l) \mapsto H^{k+1}_{loc}(\mathbb{R}^d, \Lambda^{l-1}),$$

such that for all $\omega \in H(d0, \mathbb{R}^d, \Lambda^l) \cap H^k(\mathbb{R}^d, \Lambda^l)$,

$$d\mathcal{L}\omega = \omega. \quad (32)$$

As a tool for the proof, we introduce the Fourier transform of functions, denoted by $\mathcal{F}$, mapping from $L^2(\mathbb{R}^d)$ into itself, and let $\mathcal{F}^{-1}$ stand for its inverse.

The Fourier transform (cf. [48]) of a differential l-form $\omega = \sum_I \omega_I d\mathrm{x}_I \in L^2(\mathbb{R}^d; \Lambda^l)$, still denoted by $\mathcal{F}$, is defined component wise by

$$\widehat{\omega}(\xi) := \mathcal{F}(\omega)(\xi) = \sum_I \widehat{\omega}_I(\xi) d\xi_I, \quad (33)$$

where

$$\widehat{\omega}_I(\xi) := \mathcal{F}(\omega_I)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\iota\xi \cdot \mathrm{x}) \omega_I(\mathrm{x}) d\mathrm{x} \quad (34)$$

and ι is the imaginary unit, $\xi = (\xi_1, \cdots, \xi_d)^T$ is the vectorial angular frequency in $\mathbb{R}^d$ and $d\xi_I = d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_l}$ with I being an increasing l-permutation.

Accordingly, the inverse Fourier transform of ω, also denoted by $\mathcal{F}^{-1}$, is defined by

$$\omega(\mathrm{x}) := \mathcal{F}^{-1}(\widehat{\omega})(\mathrm{x}) = \sum_I \omega_I(\mathrm{x}) d\mathrm{x}_\mathrm{I}, \quad (35)$$

where

$$\omega_I(\mathrm{x}) := \mathcal{F}^{-1}(\widehat{\omega}_I)(\mathrm{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(\iota\xi \cdot \mathrm{x}) \widehat{\omega}_I(\xi) \mathrm{d}\xi \tag{36}$$

It is easy to see that the Fourier transform converts the exterior derivative into an exterior product:

Lemma 6. For any $\omega \in H(d, \Omega, \Lambda^l)$, we have

$$\mathcal{F}(d\omega) = \iota\hat{\xi} \wedge \mathcal{F}(\omega), \tag{37}$$

where $\hat{\xi}$ is the differential 1-form in the frequency domain, namely $\hat{\xi} = \xi_1 d\xi_1 + \xi_2 d\xi_2 + \cdots + \xi_d d\xi_d$.

Proof. We follows the idea in the proof of [48]. It boils down to straightforward calculations with Fourier transforms of differential forms.

Let $\omega \in H(d0, \mathbb{R}^d, \Lambda^l) \cap H^k(\mathbb{R}^d, \Lambda^l)$, i.e., $d\omega = 0$. We try to seek a $\eta \in H^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$ such that for any compact $D \subset \mathbb{R}^d$,

$$d\eta = \omega \text{ and } \| \eta \|_{H^{k+1}(D,\Lambda^{l-1})} \leq C \| \omega \|_{\mathrm{H^k}(\mathrm{D},\Lambda^{\mathrm{l}})}. \tag{38}$$

Taking the Fourier transform on both sides of the equations $d\eta = \omega$ and $d\omega = 0$, we get from (37)

$$\iota\hat{\xi} \wedge \hat{\eta} = \widehat{\omega}, \quad \iota\hat{\xi} \wedge \widehat{\omega} = 0. \tag{39}$$

This linear system has a solution given by

$$\hat{\eta}(\xi) := \frac{-\iota\xi \lrcorner \widehat{\omega}(\xi)}{|\xi|^2}. \tag{40}$$

Thus we have

$$\iota\hat{\xi} \wedge \hat{\eta} = \sum_J \frac{\sum_{m\in J} \xi_m^2 + \mathfrak{A}}{|\xi|^2} \omega_J, \tag{41}$$

where

$$\xi \lrcorner (\hat{\xi} \wedge \widehat{\omega}) = \sum_J \omega_J(\xi \lrcorner (\hat{\xi} \wedge d\xi_J)) = \sum_J \omega_J \left(\sum_{m\notin J} \xi \lrcorner (\xi_m d\xi_m \wedge d\xi_J) \right). \tag{42}$$

Without loss of generality, we assume that $j_1 < \cdots < j_{i_m} < m < j_{i_m+1} < \cdots < j_l$ for $m \notin J$ and denote $J \cup \{m\}$ by the increasing $l+1$-permutation $\{j_1, \ldots, j_{i_m}, m, j_{i_m+1}, \ldots, j_l\}$.

Appealing to the Fourier representation of Sobolev norms on $\mathbb{R}^d$, we can conclude $\frac{\partial \eta_I}{\partial x_j} \in H^k(\mathbb{R}^d)$ for all combinations of I and j.

Next we can choose a cut-off function $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi(\boldsymbol{\xi}) = 1$ for $|\boldsymbol{\xi}| \leq 1$, and $\psi(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}| \geq 2$. Then split $\widehat{\boldsymbol{\omega}}$ according to

$$\hat{\eta}(\xi) = \psi(\xi)\hat{\eta}(\xi) + \big(1 - \psi(\xi)\big)\hat{\eta}(\xi). \qquad (43)$$

Note that each component of the differential form $\psi(\xi)\hat{\eta}(\xi)$ has a compact support and belongs to $L^1(\mathbb{R}^d)(d \geq 3!)$, so that its inverse Fourier transform is analytic. Hence, the restriction of $\mathcal{F}^{-1}(\psi(\cdot)\hat{\eta}(\cdot))$ to any compact $D \subset \mathbb{R}^d$ belongs to $H^m(D)$ for any $m \in \mathbb{N}_0$. It goes without saying that the inverse Fourier transform of the second term $(1 - \psi(\xi))\hat{\eta}(\xi)$ yields a form in $H^k(\mathbb{R}^d, \Lambda^{l-1})$ Summing up, we have shown that $\mathcal{F}^{-1}(\hat{\eta}(\cdot)) \in H^k_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$. This completes our proof.

Corollary 1. (General lifting lemma). Let $\mathrm{k} \in \mathbb{N}_0$ and $1 \leq \mathrm{l} \leq \mathrm{d}$. For a bounded Lipschitz domain $\Omega \in \mathbb{R}^{\mathrm{d}}$ of full topological generality and all $\omega \in \mathrm{d}\overline{\mathrm{H}}^{\mathrm{k}}(\mathrm{d}, \Omega, \Lambda^{\mathrm{l}-1})$, then there is a $\eta \in \mathrm{H}^{\mathrm{k}+1}(\Omega, \Lambda^{\mathrm{l}-1})$ and a positive constant C independent of η such that

$$d\eta = \omega, \qquad (44)$$

$$\| \eta \|_{H^{k+1}(\Omega,\Lambda^{l-1})} \leq C \| \omega \|_{H^k(\Omega,\Lambda^l)}. \qquad (45)$$

Moreover, for all $\omega \in \mathrm{dH}^{\mathrm{s}}_0(\mathrm{d}, \Omega, \Lambda^{\mathrm{l}-1})$ for $1 \leq \mathrm{l} < \mathrm{d}$, and $\int_\Omega \omega = 0$ if $\mathrm{l} = \mathrm{d}$, there is a $\eta \in \mathrm{H}^{\mathrm{s}+1}_0(\Omega, \Lambda^{\mathrm{l}-1})$ and a positive constant C independent of η such that (44) and (45) holds.

Proof. By Theorem 3, we prove the desired result by defining $\eta = \mathrm{R}\omega$ or $\eta = \mathrm{R}_0\omega$, which show the first and second parts, respectively.

It is natural to derive from Corollary 1 a similar result for the curl operator to Lemma 6. Assuming that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then there exists a positive constant C such that for all $\mathrm{v} \in \mathrm{curl}\,\mathrm{H}(\mathrm{curl}; \Omega)$, one can find $\mathrm{u} \in \mathrm{H}^1(\Omega)$ satisfying

$$\mathrm{curl}\,\mathrm{u} = \mathrm{v} \quad \text{and} \quad \| \mathrm{u} \|_{\mathrm{H}^1(\Omega)} \leq \mathrm{C} \| \mathrm{v} \|_{\mathrm{L}^2(\Omega)} \qquad (46)$$

if $\mathrm{v} \in \mathrm{curl}\,\mathrm{H}_0(\mathrm{curl}; \Omega)$, we can find a$\mathrm{u} \in \mathrm{H}^1_0(\Omega)$ such that (46) holds.

IV. Conclusion

This paper has delved into the construction and application of universal extension operators for Sobolev spaces of differential forms on Lipschitz-bounded domains, emphasizing the pivotal role of these spaces in addressing the intricate challenges posed by complex geometric environments. We set out to demonstrate that Sobolev spaces of differential forms, while deeply rooted in mathematical theory, are far from abstract concepts. Instead, they are indispensable tools that empower engineers and scientists to tackle real-world problems in the domains where traditional Sobolev spaces often fall short. By bridging the realms of mathematical rigor and practical engineering and scientific applications, this research establishes a valuable connection. It enables us to systematically model, analyze, and optimize complex systems and phenomena, such as structural mechanics in civil engineering or electromagnetic field analysis in electrical engineering, with mathematical precision. The universal extension operators introduced in this paper serve as versatile tools that facilitate the navigation of Lipschitz-bounded domains, and through their application, we have underscored the fundamental character of Sobolev spaces of differential forms in modern engineering and science. In the dynamic landscape of engineering and scientific exploration, the synergy of mathematical theory and practical applications is crucial. This research offers a path for engineers and scientists to combine mathematical insight with real-world problem-solving. It is a testament to the ever-evolving role of mathematics in shaping engineering and scientific progress, and it highlights the enduring importance of Sobolev spaces of differential forms in addressing the complexities of our ever-changing world.

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