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Giving the Identity of Two-Dimensional Endo-Commutative Algebras By Structure Constants

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Abstract

We examine the identity of so-called endo-commutative algebras in this study. It is evident from the character required to denote this class that the item in this class efficiently maintains the square of components. This concise and straightforward explanation makes use of a recent finding by one of the authors regarding the representation of the category of all two-dimensional algebras over any elementary field.

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Introduction

One of the key and challenging algebraic issues is the classification problem, up to isomorphism, of a given class of algebras. There are currently two methods known to solve the issue. The structural (base free, invariant) method is one of them. Examples of this type of approach are the classifications of simple and semi-simple Lie algebras by Cartan and of finite dimensional simple and semi-simple associative algebras by Wedderburn. However, it has been noted that this method gets more challenging when more generic algebraic types are taken into account.

Recently, in [5] a complete classification of all 2-dimensional algebras over any basic field has been announced and we have come across with the paper [7], where the authors gave a complete classification of 2-dimensional endo-commutative algebras over the field $\mathbb{Z}_2$. We have realized that it can be done over an arbitrary field by more easy and faster way relying on the results of [5] This is the subject of the present paper.

PRELIMINARIES

We start by quickly reviewing a few of the ideas that will be used in the article. Let **A** be an n-dimensional algebra over a field $\mathbb{F}$ and $\mathbf{e} = \{e_1, e_2, \dots, e_n\}$ be a basis of **A**. Then on the basis **e** is represented by a $n \times n^2$ matrix (called the matrix of structure constant, shortly MSC)

$$A = \begin{pmatrix} a_{11}^1 & a_{12}^1 & \dots & a_{1n}^1 & a_{21}^1 & a_{22}^1 & \dots & a_{2n}^1 & \dots & a_{n1}^1 & a_{n2}^1 & \dots & a_{nn}^1 \\ a_{11}^2 & a_{12}^2 & \dots & a_{1n}^2 & a_{21}^2 & a_{22}^2 & \dots & a_{2n}^2 & \dots & a_{n1}^2 & a_{n2}^2 & \dots & a_{nn}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{11}^n & a_{12}^n & \dots & a_{1n}^n & a_{21}^n & a_{22}^n & \dots & a_{2n}^n & \dots & a_{n1}^n & a_{n2}^n & \dots & a_{nn}^n \end{pmatrix}$$

as follows

$$e_i \cdot e_j = \sum_{k=1}^{n} a_{ij}^k e_k, \text{ where } i, j = 1,2, \dots, n.$$

Therefore, the product on **A** with respect to the basis **e** is written as follows

(2.1) $$x \cdot y = \mathbf{e}A(x \otimes y)$$

for any $x = \mathbf{e}x, y = \mathbf{e}y$, where $x = (x_1, x_2, \dots, x_n)^T$, and $y = (y_1, y_2, \dots, y_n)^T$ are column coordinate vectors of x and y, respectively, $x \otimes y$ is the tensor(Kronecker) product of the vectors x and y. Now and onward for the product "x · y" on **A** we use the juxtaposition "xy".

If **A** is a two-dimensional algebra over a field $\mathbb{F}$ and $\mathbf{e} = \{e_1, e_2\}$ is a linear basis then MSC of it looks like

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

GIVING THE INDENTITY OF TWO-DIMENTIONAL ENDO-COMMUTATIVE ALGEBRAS BY STRUCTURE CONSTANTS

Definition 1. *An algebra* **A** *is said to be endo-commutative if* $x^2y^2 = (xy)^2$, *for any* $x, y \in$ **A**. According to (2.1) we write

$$x^2 = \mathbf{e}Ax^{\otimes 2}, \quad y^2 = \mathbf{e}Ay^{\otimes 2}, \quad x^2y^2 = \mathbf{e}A\left(Ax^{\otimes 2} \otimes Ay^{\otimes 2}\right) = \mathbf{e}A(A \otimes A)\left(x^{\otimes 2} \otimes y^{\otimes 2}\right)$$

$$(xy)^2 = \mathbf{e}A\big(A(x \otimes y) \otimes A(x \otimes y)\big) = \mathbf{e}A(A \otimes A)(x \otimes y)^{\otimes 2}.$$

Therefore, the equality $x^2y^2 = (xy)^2$ in terms of MSC and the coordinate vectors is written as follows

$$\mathbf{e}A(A \otimes A)\left(x^{\otimes 2} \otimes y^{\otimes 2}\right) = \mathbf{e}A(A \otimes A)(x \otimes y)^{\otimes 2}$$

$$A(A \otimes A)\left(x^{\otimes 2} \otimes y^{\otimes 2} - (x \otimes y)^{\otimes 2}\right) = 0$$

or just

(3.1) $$A(A \otimes A)(x \otimes x \otimes y \otimes y - x \otimes y \otimes x \otimes y) = 0.$$

Let now **A** be a two-dimensional algebra over a field $\mathbb{F}$ and let

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$$

be its MSC on a basis $\mathbf{e} = \{e_1, e_2\}$.

Then (3.1) is nothing than

$$\begin{cases} A_3 - A_5 & = & 0 \\ A_4 - A_6 & = & 0 \\ A_6 - A_4 + A_{11} - A_{13} & = & 0 \\ A_{12} - A_{14} & = & 0 \\ A_{13} - A_{11} & = & 0 \\ B_3 - B_5 & = & 0 \\ B_4 - B_6 & = & 0 \\ B_6 - B_4 + B_{11} - B_{13} & = & 0 \\ B_{12} - B_{14} & = & 0 \\ B_{13} - B_{11} & = & 0 \end{cases} \qquad (3.2)$$

where

$A_3 = \alpha_1^2\alpha_3 + \alpha_1\alpha_2\beta_3 + \alpha_3^2\beta_1 + \alpha_4\beta_1\beta_3,$
$A_4 = \alpha_1^2\alpha_4 + \alpha_1\alpha_2\beta_4 + \alpha_3\alpha_4\beta_1 + \alpha_4\beta_1\beta_4,$
$A_5 = \alpha_1^2\alpha_2 + \alpha_2^2\beta_1 + \alpha_1\alpha_3\beta_2 + \alpha_4\beta_1\beta_2,$
$A_6 = \alpha_1\alpha_2^2 + \alpha_2^2\beta_2 + \alpha_2\alpha_3\beta_2 + \alpha_4\beta_2^2,$
$A_{11} = \alpha_1\alpha_3^2 + \alpha_2\alpha_3\beta_3 + \alpha_3^2\beta_3 + \alpha_4\beta_3^2,$
$A_{12} = \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\beta_4 + \alpha_3\alpha_4\beta_3 + \alpha_4\beta_3\beta_4,$
$A_{13} = \alpha_1^2\alpha_4 + \alpha_2\alpha_4\beta_1 + \alpha_1\alpha_3\beta_4 + \alpha_4\beta_1\beta_4,$
$A_{14} = \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_4\beta_2 + \alpha_2\alpha_3\beta_4 + \alpha_4\beta_2\beta_4,$
$B_3 = \alpha_1\alpha_3\beta_1 + \alpha_1\beta_2\beta_3 + \alpha_3\beta_1\beta_3 + \beta_1\beta_3\beta_4,$
$B_4 = \alpha_1\alpha_4\beta_1 + \alpha_1\beta_2\beta_4 + \alpha_4\beta_1\beta_3 + \beta_1\beta_4^2,$
$B_5 = \alpha_1\alpha_2\beta_1 + \alpha_2\beta_1\beta_2 + \alpha_1\beta_2\beta_3 + \beta_1\beta_2\beta_4,$
$B_6 = \alpha_2^2\beta_1 + \alpha_2\beta_2^2 + \alpha_2\beta_2\beta_3 + \beta_2^2\beta_4,$
$B_{11} = \alpha_3^2\beta_1 + \alpha_3\beta_2\beta_3 + \alpha_3\beta_3^2 + \beta_3^2\beta_4,$
$B_{12} = \alpha_3\alpha_4\beta_1 + \alpha_3\beta_2\beta_4 + \alpha_4\beta_3^2 + \beta_3\beta_4^2,$
$B_{13} = \alpha_1\alpha_4\beta_1 + \alpha_4\beta_1\beta_2 + \alpha_1\beta_3\beta_4 + \beta_1\beta_4^2,$
$B_{14} = \alpha_2\alpha_4\beta_1 + \alpha_4\beta_2^2 + \alpha_2\beta_3\beta_4 + \beta_2\beta_4^2.$

Note that the set of the functions $\{x_1^2y_1y_2, x_1^2y_2^2, x_1x_2y_1^2, x_1x_2y_1y_2, x_1x_2y_2^2, x_2^2y_1^2, x_2^2y_1y_2\}$ is linearly independent. Therefore, the system (3.2) in terms of A_i and B_j $(i, j = 3,4,\ldots,13)$ can be rewritten as follows
(3.3)

$$\begin{cases} A_3 - A_5 & = & 0 \\ A_4 - A_6 & = & 0 \\ A_6 - A_4 + A_{11} - A_{13} & = & 0 \\ A_{12} - A_{14} & = & 0 \\ A_{13} - A_{11} & = & 0 \\ B_3 - B_5 & = & 0 \\ B_4 - B_6 & = & 0 \\ B_6 - B_4 + B_{11} - B_{13} & = & 0 \\ B_{12} - B_{14} & = & 0 \\ B_{13} - B_{11} & = & 0 \end{cases} \qquad (3.3)$$

Since equations 3 and 8 in the system rely linearly on the other equations, they can be eliminated. In terms of the structure constants α_i, β_j $(i, j = 1,2,\ldots,4)$ the system (3.3) is written as follows

$$
\begin{cases}
(\alpha_1^2\alpha_3 + \alpha_1\alpha_2\beta_3 + \alpha_3^2\beta_1 + \alpha_4\beta_1\beta_3) - (\alpha_1^2\alpha_2 + \alpha_2^2\beta_1 + \alpha_1\alpha_3\beta_2 + \alpha_4\beta_1\beta_2) & = & 0 \\
(\alpha_1^2\alpha_4 + \alpha_1\alpha_2\beta_4 + \alpha_3\alpha_4\beta_1 + \alpha_4\beta_1\beta_4) - (\alpha_1\alpha_2^2 + \alpha_2^2\beta_2 + \alpha_2\alpha_3\beta_2 + \alpha_4\beta_2^2) & = & 0 \\
(\alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\beta_4 + \alpha_3\alpha_4\beta_3 + \alpha_4\beta_3\beta_4) - (\alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_4\beta_2 + \alpha_2\alpha_3\beta_4 + \alpha_4\beta_2\beta_4) & = & 0 \\
(\alpha_1^2\alpha_4 + \alpha_2\alpha_4\beta_1 + \alpha_1\alpha_3\beta_4 + \alpha_4\beta_1\beta_4) - (\alpha_1\alpha_3^2 + \alpha_2\alpha_3\beta_3 + \alpha_3^2\beta_3 + \alpha_4\beta_3^2) & = & 0 \\
(\alpha_1\alpha_3\beta_1 + \alpha_1\beta_2\beta_3 + \alpha_3\beta_1\beta_3 + \beta_1\beta_3\beta_4) - (\alpha_1\alpha_2\beta_1 + \alpha_2\beta_1\beta_2 + \alpha_1\beta_2\beta_3 + \beta_1\beta_2\beta_4) & = & 0 \\
(\alpha_1\alpha_4\beta_1 + \alpha_1\beta_2\beta_4 + \alpha_4\beta_1\beta_3 + \beta_1\beta_4^2) - (\alpha_2^2\beta_1 + \alpha_2\beta_2^2 + \alpha_2\beta_2\beta_3 + \beta_2^2\beta_4) & = & 0 \\
(\alpha_3\alpha_4\beta_1 + \alpha_3\beta_2\beta_4 + \alpha_4\beta_3^2 + \beta_3\beta_4^2) - (\alpha_2\alpha_4\beta_1 + \alpha_4\beta_2^2 + \alpha_2\beta_3\beta_4 + \beta_2\beta_4^2) & = & 0 \\
(\alpha_1\alpha_4\beta_1 + \alpha_4\beta_1\beta_2 + \alpha_1\beta_3\beta_4 + \beta_1\beta_4^2) - (\alpha_3^2\beta_1 + \alpha_3\beta_2\beta_3 + \alpha_3\beta_3^2 + \beta_3^2\beta_4) & = & 0
\end{cases} \tag{3.4}
$$

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