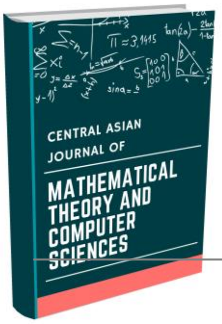

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ON FIXED POINTS OF STRICTLY POSITIVE NONLINEAR STOCHASTIC OPERATORS ON A ONE-DIMENSIONAL SIMPLEX

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Abstract

In this paper, we consider the fixed points of strictly positive nonlinear stochastic operators on the one-dimensional simplex. Theorems for the number of fixed points of strictly positive nonlinear stochastic operators are proved.

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Nonlinear equations arise in many problems of modern mathematical physics, genetics and technology. A special role in this case is played by the theory of fixed points of nonlinear operators. It is known that the number and location of fixed points, cycles and the limiting behavior of orbits determine the qualitative behavior of a dynamic system. Note that interest in the theory of nonlinear stochastic operators on the simplex is due to its relevance in problems of population genetics [1]. There are numerous publications on the study of the properties and characters of fixed points of quadratic stochastic operators defined on the simplex [2-3]. The works [4-6] are devoted to the study of cubic operators on a finite-dimensional simplex.

In this paper we study fixed points of strictly positive stochastic operator of degree four on a one-dimensional simplex.

Let $E=\{1,2,3,...,n\}$. By the $(m-1)$-simplex we mean the set

$$S^{m-1}=\left\{x=(x_1,x_2,...,x_m)\in R^m : x_i\geq 0, \forall i\in E, \sum_{i=1}^{m} x_i=1\right\}.$$

Let us denote by $S_{>}^{m-1}$ inside of the simplex S^{m-1}, i.e.

$$S_{>}^{m-1}=\left\{x=(x_1,x_2,...,x_m)\in R^m : x_i> 0, \forall i\in E, \sum_{i=1}^{m} x_i=1\right\}.$$

Each element $x\in S^{m-1}$ is a probability measure on E, and so it may be looked upon as the state of a biological (physical and so on) system of m elements.

Definition 1 [7]. *An arbitrary continuous operator C_4 defined on the simplex S^{m-1} will be called stochastic if*

$$\left(C_n x\right)_k = x'_k = \sum_{i_1,i_2,\ldots,i_n=1}^{m} P_{i_1 i_2 \ldots i_n,k} x_{i_1} x_{i_2} \ldots x_{i_n}, \qquad \forall k=\overline{1,m},$$

where

$$P_{i_1 i_2 \ldots i_n,k} > 0, \quad \forall i_j = \overline{1,m}, \quad j=\overline{1,n}, \quad k=\overline{1,m};$$

$$P_{i_1 i_2 \ldots i_n,k} = P_{i_{\pi(1)} i_{\pi(2)} \ldots i_{\pi(n)},k}, \qquad k=\overline{1,m},$$

for every permutation π *and*

$$\sum_{k=1}^{m} P_{i_1 i_2 \ldots i_n,k} = 1, \quad \forall i_j = \overline{1,m}, \quad j=\overline{1,n}.$$

Recall that the stochastic operator maps the simplex onto itself. In Definition 1, the number n is called the order (degree) of the stochastic operator. When *n=1* operator C_n is called the linear stochastic operator, when *n=2* it is called the quadratic stochastic operator, and when *n=3* it is called the cubic stochastic operator.

IN THE CASE $n=4, m=2$

We consider following $C_4 : R^2 \to R^2$ strictly positive stochastic operator of degree four in the domain of S^1 :

$$C_4\left(x_1,x_2\right) = \left(\sum_{i,j,k,l=1}^{2} P_{ijkl,1} x_i x_j x_k x_l, \ \sum_{i,j,k,l=1}^{2} P_{ijkl,2} x_i x_j x_k x_l \right)$$

where

$$P_{1112,m} = P_{1121,m} = P_{1211,m} = P_{2111,m}, \ P_{1122,m} = P_{1221,m} = P_{1212,m} = P_{2121,m} = P_{2112,m} = P_{2211,m},$$

$$P_{2221,m} = P_{2112,m} = P_{2122,m} = P_{1222,m}, \ \ 0 < P_{ijkl,m} < 1, \ \sum_{m=1}^{2} P_{ijkl,m} = 1, \ \forall i,j,k,l,m \in E = \{1,2\}.$$

The problem considered in this paper is to find sufficient conditions for the number of fixed points in the one-dimensional simplex of the stochastic operator C_4.

Denote by $FixC_4$ the set of all fixed points of the operator C_4, i.e.

$$FixC_4 = \left\{ \omega \in S^1 : C_4 \omega = \omega \right\}.$$

We put following:

$$\mu_0 = P_{1111,1} - 4P_{1112,1} + 6P_{1122,1} - 4P_{1222,1} + P_{2222,1},$$

$$\mu_1 = 4P_{1112,1} - 12P_{1122,1} + 12P_{1222,1} - 4P_{2222,1}, \ \mu_2 = 6P_{1122,1} - 12P_{1222,1} + 6P_{2222,1},$$

$$\mu_3 = 4P_{1222,1} - 4P_{2222,1} - 1, \ \mu_4 = P_{2222,1}$$

and denote polynomial of order four, i.e.

$$P_4\left(x\right) = \mu_0 x_1^{\ 4} + \mu_1 x_1^{\ 3} + \mu_2 x_1^{\ 2} + \mu_3 x_1 + \mu_4.$$

Lemma 1. *The number of roots the polynomial* $P_4(x)$ *on* $(0,1)$ *is equal to the number of fixed point of the stochastic operator* C_4 *on* S^1.

Proof. It is easy to see that for all $\omega_0=(x_1^0,x_2^0)\in FixC_4$ we have $C_4\omega_0=\omega_0$. We get $P_4(x_1^0)=0$ by using equality $x_1^0+x_2^0=1$, i.e., the point x_1^0 is root of the polynomial $P_4(x)$ on $(0,1)$.

Let the point x_0 is root of the polynomial $P_4(x)$ on $(0,1)$. Then the point $\omega_0=(x_0,1-x_0)$ is solution of the equation $C_4\omega=\omega$.

Lemma 2. *The polynomial* $P_4(x)$ *has at least one root in the interval* $(0,1)$.

Proof. It is easy to see that the inequalities hold $P_4(0)=\mu_4=P_{2222,1}>0$, $P_4(1)=P_{1111,1}-1=-P_{1111,2}<0$. Therefore, polynomial $P_4(x)$ has at least one point on $(0,1)$.

Consequence 1. *Operator* C_4 *has at least one fixed point in* S^1, *i.e.* $|FixC_4|\geq 1$ *(where* $|A|$ *-* A *is the power of the set).*

Let us introduce the following notation:

$$p=\frac{3\mu_1^2}{16\mu_0^2}-\frac{\mu_2}{2\mu_0}, \qquad q=-\frac{\mu_1\mu_2}{8\mu_0^2}+\frac{\mu_3}{4\mu_0}+\frac{\mu_1^3}{32\mu_0^3},$$

$$Q=\left(\frac{p}{3}\right)^3+\left(\frac{q}{2}\right)^2.$$

For $Q<0$ we define the following values:

$$\lambda_k=2\sqrt{-\frac{p}{3}}\cos\left(\frac{\varphi+2\pi(k-2)}{3}\right)-\frac{\mu_1}{4\mu_4},\ k=1,2,3,$$

$$\cos\varphi=-\frac{q}{2}\left(-\frac{3}{p}\right)^{\frac{3}{2}},\ 0\leq\varphi\leq\pi.$$

For convenience, we introduce the following notation:

$\alpha=\lambda_3,\ \beta=\lambda_1,\ \gamma=\lambda_2.$

RESULTS

For the case $\mu_0>0$, theorems are given for the number of fixed points of the operator C_4 in S^1.

Theorem 1. *Let* $\mu_0>0$, $Q<0$, $\alpha>0$, $\gamma<1$ *be satisfied. If for the polynomial* $P_4(x)$ *satisfy the following properties*

(a) $P_4(\alpha)>0,$

(b) $P_4(\beta)<0,$

then operator C_4 *has unique fixed point in* S^1*, i.e.* $|FixC_4|=1$.

Proof. Let $\mu_0>0$, $Q<0$ and $\alpha>0$, $\gamma<1$. It is known that by the property $\mu_0>0$, we have $P_4(\pm\infty)=+\infty$. By the property $Q<0$, the numbers α, β and γ are different solutions of the equation $P_4'(x)=0$. This solutions are extreme points of the function $y=P_4(x)$. Moreover by definition of the α, β and γ, we take this $0<\alpha<\beta<\gamma<1$ [8]. By the global monotonicity function $y=P_4(x)$, we have $\min\limits_{x\in[0;\beta]}=P_4(\alpha)$, $\min\limits_{x\in[\beta;1]}P_4(x)=P_4(\gamma)$ and $\max\limits_{x\in[\alpha;\gamma]}P_4(x)=P_4(\beta)$. By Lemma 1, to find the number of fixed points of the operator C_4 in S^1, it is enough to check the roots of the polynomial $y=P_4(x)$ in $(0,1)$.

(a) Let $P_4(\alpha)>0$. Then we have $P_4(\beta)>0$. According to inequality $P_4(1)<0$, it turns out that the value of function $y=P_4(x)$ at point γ is negative, i.e. $P_4(\gamma)<0$. Thus, $y=P_4(x)$ function intersect the Ox axis at one point at (β,γ) intervals.

(b) Let $P_4(\beta)<0$. We know that $P_4(0)>0$, $P_4(1)<0$ and $\max\limits_{x\in[\alpha;\gamma]}P_4(x)=P_4(\beta)$. Then function $y=P_4(x)$ has unique root in $(0,\alpha)$.

REFERENCES

1. Любич Ю.И.: Математические структуры в популяционной генетике. *Киев: Наук. Думка,* 1983.
2. Ganikhodzhaev R., Mukhamedov F., Rozikov U.: Quadratic stochastic operators and processes results and open problems, *Infinite Dimensional Analysis Quantum Probability and Related Topics*, **14**:2 (2011) 279-335.
3. Mukhamedov F., Ganixodjaev N.: Quantum quadratic operators and prosses, *Lecture Notes in mathematics book*, 2133, Nov. 12, 2015.
4. Розиков У.А., Хамроев А.Ю.: О кубических операторах определенных в конечномерном симплексе, *Укр. мат. журн.,* 2004. Т. 56, № 10. С. 1424-1433.
5. Jamilov U.U., Khamraev A.Yu., Ladra M.: On a Volterra Cubic Stochastic Opearator, *Bull Math Biol*, **80**:2 (2018) 319-334.
6. Нодиров Ш.Д. О неподвижных точках строго положительных кубических стохастических операторов на одномерном симплексе, *Илм сарчашмалари УрГУ*, Научно-методический журнал, 2019, № 12, С.10-16.
7. Шахиди Ф.А.: О биостохастических операторах, определенных в конечномерном симплексе, *Сибирский математический журнал,* Март-апрель, 2009.Том. 50, № 2. С. 463-468.
8. Nickalls, R.W.D.: Vieta, Descartes and the cubic equation. *Mathematical Gazette*, **90**(July 2006), 203-208.