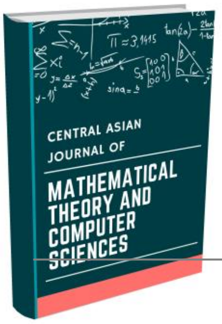

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A NEW TECHNIQUE TO SOLVE PREDATOR–PREY MODELS BY USING SHEHU TRANSFORMATION-AKBARI-GANJI'S METHOD WITH PADÉ APPROXIMANT

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Abstract

In this paper, a new technique has been proposed and applied to find analytical approximate solutions for predator-prey systems. The new technique depends on combining the algorithms of Shehu transform and Akbari-Ganji's method (AGM) with Padé approximant. Three examples are given to test the effectiveness, accuracy, and performance of the suggested method. In comparison to the existing methods used to identify the analytical approximate solutions of the current problems, the results of using the new technique demonstrate that it has excellent efficiency and accuracy. Also, the tables and errors graphs show the applicability and necessity of this technique.

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1. Introduction

The primary model for analyzing the dynamics of many species' populations is the predator-prey model, which was developed by Lotka and Volterra [1, 2]. It has a broad range of applications in many different scientific fields, including chemical processes [3, 4], bioparticle granulation [5], and the interaction of microorganisms and ecosystems [6, 7]. The Lotka-Volterra type predator-prey system has been the focus of a lot of research in recent years. Zhu and Yin focused on the competitive Lotka-Volterra model in random situations [8]. Badri et al. focus on installing Lotka-Volterra systems, a particular family of nonlinear quadratic systems, and possible balance points [9]. This paper's objective is to numerically study the predator-prey models controlled by the following system of nonlinear ordinary differential equations

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x(t), y(t), a_1(t), \dots, a_n(t)) \\ \frac{dy}{dt} &= g(x(t), y(t), b_1(t), \dots b_n(t)) \end{aligned}\right\} \qquad (1)$$

where the positive functions $a_i(t)$, b_i (t) typically give relative measurements of the effect of dimensional parameters [10] and $x(t), y(t)$, respectively, reflect the population densities of prey and predator at time t. The numerical solution of system (1) is not straightforward due to the structure of the functions f and g, hence it is essential to create trustworthy numerical algorithms. Numerous articles have addressed the numerical solution of predator-prey models to compare the effectiveness and reliability of various approaches numerically. For instance, the Adomian technique has been quantitatively tested using a predator-prey model [11-14]. Additionally, He's variational approach was researched and used to simulate a predator-prey relationship [15]. The predator-prey model has also been used using nonstandard finite difference techniques [16]. The differential transformation method (DTM) was used to analyze a predator-prey model with constant coefficients over a brief time horizon [17]. The ratio-dependent predator-prey system with continuous effort harvesting was approximated using the homotopy perturbation approach [18]. The nonlinear imprecise prey-predator model with stability analysis was implemented using the homotopy perturbation and variational iteration methods [19]. Many researchers have presented studies on the characteristics of prey and predator systems such as coexistence, stability and extinction [20-22]. A numerical study is presented in a ratio-dependent predatory system disturbed by time noise. The base model was calculated using two charts. Euler forward diagram and finite difference diagram [23]. Using a Holling type-II predator functional response, a reaction-diffusion-advection predator-prey model was shown to be stable [24]. A multispecies ecological and epidemiological mathematical model was designed for scenarios where interacting species compete for the same food sources and the prey are infected. The stability of the aforementioned method is examined by the Von Neumann stability analysis [25]. What was presented above reflects the importance of studying the system and the importance of treating it with different simulation methods. In general, in many cases, it is difficult to find a solution to nonlinear differential equations using the methods of integrative transformations due to the nonlinear terms in them. Moreover, according to our information, the process of merging analytical methods and integrative transformations leads to a reduction in the number of computational operations and reduces the difficulty of analytical methods are used alone. Many studies have focused on this area [26-29]. Many analytical methods used to solve differential equations, and one of these methods is Akbari-Ganji's method developed by Ganji and Akbari and could be used to solve a variety of nonlinear ordinary and partial differential equations. It is characterized by its ease of application to find solutions [30]. In addition, Pade´s approximation is the best approximation of a function near a certain point by a logical function of a specific order. It improves accuracy and the convergence of solutions by improving the field, which was developed by Henri Pade' in 1890 [31]. A recent integrative transformation is the Shehu transform, which was developed by Maitama and Zhao in 2019 [32]. And the fact that these aforementioned methods have not been used before to solve the prey and predator systems. Consequently, this has prompted us to propose a new method. This new method combines the algorithms of the Shehu transform method, Akbari- Ganji's method, and Pade´s approximation method (SAGPM). This paper involves seven sections, of which this introduction is the first. In Section 2, We present the proposed technical algorithm, and we applied it to different systems of predator- prey models (Section 3). Next, we introduce the numerical results and compare these results with other works (Section 4). In Sections 5 and 6, we study the convergence and stability analysis of predator-prey systems. The last Section summarizes the major findings of this study. When applying the proposed approach and comparing it with the previous works that dealt with the same problem, the results have shown the effectiveness and efficiency of the proposed approach in terms of high accuracy and good convergence.

2. The new SAGPM Algorithm

The basic idea of the SAGPM is based on the Shehu transform method and Akbari-Ganji's method with Padé approximate algorithms, which will be mentioned in this section.

2.1. Shehu transformation method [32]:

Shehu transform, a new integral transformation that Shehu and Zhao introduced in 2019, is a generalization of the Laplace and Sumudu integral transforms. The authors have used it to resolve both the ordinary and partial differential equations.

The Shehu transform is found across set A:

$$A = \{ f(t) : \exists\, N, n_1, n_2 > 0, |f(t)| < N\, exp\left(\frac{|t|}{n_j}\right), if\ t \in \left(-1\right)^j \times [0, \infty) \},$$

by the following integral

$S[f(t)] = F(v,u) = \int_0^\infty exp(\frac{-vt}{u})\, f(t)\, dt$

$= lim_{a\to\infty} \int_0^a exp(\frac{-vt}{u})\, f(t)\, dt, \quad v > 0, u > 0.$ (2)

The inverse Shehu transform is given by

$S^{-1}[F(v,u)] = f(t), \quad for\ t \geq 0.$

Equivalently

$f(t) = S^{-1}[F(v,u)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{u} exp(\frac{vt}{u})\, F(v,u)\, dv,$ (3)

where v and u are the variables of the Shehu transform, and α is the real constant and the integral in Eq. (3) is taken along v = α in the complex plane $v = x + iy$.

2.1.1. Shehu transformation of derivatives and some functions

If $S\{f(t)\} = F(v,u)$ then

1) $S\left\{\frac{df}{dt}\right\} = \frac{v}{u}F(v,u) - f(0)$

2) $S\{f^{(n)}(t)\} = \frac{v^n}{u^n}F(v,u) - \sum_{k=0}^{n-1}\left(\frac{v}{u}\right)^{n-(k+1)} f^{(k)}$

3) $S\{t\} = \frac{v^2}{u^2}$

4) $S\{\exp(at)\} = \frac{v}{u-av}$

5) $S\{\cos(t)\} = \frac{vu}{u^2+v^2}$

6) $S\{\sin(t)\} = \frac{vu}{u^2-v^2}$

7) $S\left\{\frac{t^n}{n!}\right\} = \left(\frac{v}{n}\right)^{n+1}, \qquad n = 0,1,2, \ldots$

2.2. Akbari-Ganji's method (AGM)

This method, which was developed by Akbari and Ganji, is an excellent computation methodology that may be utilized to solve various nonlinear differential equations. In this method, the solution is taken to be a finite series, hence the answer is obtained by resolving a sequence of algebraic problems.

The differential equation for a function $x(t)$ and its derivatives can be written as follows in order to use AGM:

In equation $p_k =$ f $(x, x', x'', \ldots, x^{(m)}) = 0$, $x = x\ (t)$ (4)

p_k is the nonlinear differential equation of m^{th}order derivatives with boundary conditions:

At t=0, $x(t)=x_0$, x '(t)= $x_1, \ldots, x^{(m-1)}$(t) = x_{m-1} (5a)

At t=L, $x(t)= x_{L_0}$, x '(t)= $x_{L_1}, \ldots, x^{(m-1)}$ (t)= $x_{L_{m-1}}$ (5b)

The differential equation's solution is taken into consideration to solve Eq. (4) in relation to the conditions (5a and 5b);

$x(t)=\sum_{i=0}^{n} a_i\, t^i$ (6)

Eq. (6) can be solved with high accuracy by selecting more terms. If the series (6) has (n) degrees, then there are $(n+1)$ unknown coefficients that must be determined to find the solution to the differential Eq. (4). For Eq. (6), the boundary conditions (5a) and (5b) are applied as follows:

We have the following at t=0.

$$\begin{cases} x(0) = a_0 = x_0 \\ x'(0) = a_1 = x_1 \\ x''(0) = a_2 = x_2 \\ \vdots \end{cases} \tag{7}$$

when t = L

$$\begin{cases} x(L) = a_0 + a_1 L + \dots + a_n L^n = x_{L_0} \\ x'(L) = a_1 + 2a_2 L + \dots + n a_n L^{n-1} = x_{L_1} \\ x''(L) = 2a_2 + 6a_3 L \dots + n(n-1) a_n L^{n-2} = x_{L_2} \\ \vdots \end{cases} \tag{8}$$

After substituting Eq. (6) into Eq. (4), and after applying the boundary conditions to it, we obtain:

$$\begin{aligned} p_0 &= f(x(0), x'(0), x''(0), \dots, x^{(m)}(0)) \\ p_1 &= f(x(L), x'(L), x''(L), \dots, x^{(m)}(L)) \\ &\vdots \end{aligned} \tag{9}$$

Application of the boundary conditions on the derivatives of the differential Eq. (9) is:

$$p'_k : \begin{cases} f(x'(0), x''(0), x'''(0), \dots, x^{(m+1)}(0)) \\ f(x'(L), x''(L), x'''(L), \dots, x^{(m+1)}(L)) \end{cases} \tag{10}$$

$$p''_k : \begin{cases} f(x''(0), x'''(0), x''''(0), \dots, x^{(m+2)}(0)) \\ f(x''(L), x'''(L), x''''(L), \dots, x^{(m+2)}(L)) \end{cases} \tag{11}$$

Equations from (7) to (11), $(n+1)$ equations may be worked out, and so $(n+1)$ unknown coefficients of Eq. (6), such as a_0 , a_1 , a_2 ,..., and a_n can be computed. By locating the coefficients of Eq. (6), the solution to Eq. (3) will be achieved.

2.3. Padé approximant

George Frobenius, who introduced the concept and researched the characteristics of the rational power approximation, is credited with creating the Padé approximant, a specific and classical sort of rational approximation. By expressing it as the quotient of two polynomials with various degrees, Henri Padé made substantial contributions about 1890. In comparison to truncating the Taylor series, it provides a better approximation of the function, especially in cases where the Taylor series does not converge and contains poles. This makes it superior to the Taylor series expansion. The approximation has been widely used in computer science to determine time delays[33-36].

Padé approximation is a ratio of two polynomials that come from the Tayler series expansion of a function $x(t)$ and define as [31]

$$P_m^l = \frac{\sum_{n=0}^{l} a_n t^n}{\sum_{n=0}^{m} b_n t^n} \tag{12}$$

where $b_0 = 1$.

The function $x(t)$ written by

$$x(t) = \sum_{n=0}^{\infty} c_n t^n \tag{13}$$

Also , $x(t) - P_m^l = o(t^{l+m+1})$

thus,

$$\sum_{n=0}^{\infty} c_n t^n = \frac{\sum_{n=0}^{l} a_n t^n}{\sum_{n=0}^{m} b_n t^n} \tag{14}$$

or

$$c_0 + c_1 t^1 + c_2 t^2 + \cdots = \frac{a_0 + a_1 t^1 + a_2 t^2 + \cdots}{1 + b_1 t^1 + b_2 t^2 + \cdots}$$

from Eq. (14), obtain the following system equations

$$a_0 = c_0$$
$$a_1 = c_1 + c_0 b_1$$
$$a_2 = c_2 + c_1 b_1 + c_0 b_2$$
$$\vdots$$

solve the above system of equations for a_n and b_n let the numerator degree to l and the denominator degree to m.

$$a_0 = c_0$$
$$a_1 = c_1 + c_0 b_1$$
$$a_2 = c_2 + c_1 b_1 + c_0 b_2$$
$$\vdots$$
$$a_l = c_l + c_{l-1} b_1 + c_{l-2} b_2 + \cdots + c_0 b_l$$
$$0 = c_{l+1} + c_l b_1 + c_{l-1} b_2 + \cdots c_{l-m+1} b_m$$
$$0 = c_{l+2} + c_{l+1} b_1 + c_l b_2 + \cdots + c_{l-m+2} b_m$$
$$\vdots$$
$$0 = c_{l+m} + c_{l+m-1} b_1 + c_{l+m-2} b_2 + \cdots + c_l b_m$$

Therefore; the fundamental notion of the new technique SAGPM for system (1) with initial conditions $x(0) = \alpha$, $y(0) = \beta$ is summarized according to the following steps:

Step 1: Taking Shehu transformation on both sides of (1), to get:

$$S\left(\frac{dx}{dt}\right) = S(f(x(t), y(t), a_1(t), \ldots, a_n(t))),$$
$$S\left(\frac{dy}{dt}\right) = S(g(x(t), y(t), b_1(t), \ldots b_n(t))),$$

using the differentiation property of Shehu transformation and the above initial conditions, we have:

$$X(u\,,\,v) = \frac{\alpha\, u}{v} + \frac{u}{v} S(f(x(t), y(t), a_1(t), \ldots, a_n(t))),$$
$$Y(u\,,\,v) = \frac{\beta\, u}{v} + \frac{u}{v} S(g(x(t), y(t), b_1(t), \ldots b_n(t))), \tag{15}$$

Step 2: Applying the inverse Shehu transformation on both sides of (15), to find out:

$x(t) = \alpha + S^{-1}(\frac{u}{v}S(f(x(t),y(t),a_1(t),\ldots,a_n(t)))),$

$y(t) = \beta + S^{-1}(\frac{u}{v}S(g(x(t),y(t),b_1(t),\ldots b_n(t)))),$ (16)

Step 3: Considering by the AGM as polynomials series with constant coefficients as follows:

$x(t) = \sum_{i=0}^{n} a_i\, t^i \quad , \quad y(t) = \sum_{i=0}^{n} b_i\, t^i$ (17)

After substituting (17) into (3), Eq.(16) becomes as follows:

$\sum_{i=0}^{n} a_i\, t^i = \alpha + S^{-1}(\frac{u}{v}S(f(\sum_{i=0}^{n} a_i\, t^i, \sum_{i=0}^{n} b_i\, t^i, a_1(t),\ldots,a_n(t)))),$

$\sum_{i=0}^{n} b_i\, t^i = \beta + S^{-1}(\frac{u}{v}S(g(\sum_{i=0}^{n} a_i\, t^i, \sum_{i=0}^{n} b_i\, t^i\ , b_1(t),\ldots b_n(t)))),$ (18)

We apply the boundary conditions to obtain some values of the coefficients, and by continuing the derivation of the Eq. (18) and substituting the boundary conditions, we get the rest of the values a_i *and* b_i.

Step 4: Applying the Padé approximation of an order $[l\ /\ m]$ on a power series solution obtained by using (SAGPM). The values l and m are arbitrarily selected. In this stage, we get the final solution.

3. Numerical experiments

In this part, three different systems corresponding to predator-prey models are solved using the new technique.

3.1. Example 1:

The first model depicts the problem of some rabbits and foxes coexisting, with foxes preying on the bunnies and rabbits preying on clover, and with an increase and reduction in the number of foxes and rabbits, respectively [37]. The ordinary differential equation system shown below serves as the model's analytical representation:

$$\left.\begin{array}{ll} \dot{x}_1(t) = \alpha_1\ x_1(t) - \alpha_2\ x_1(t)x_2(t)\ , & x_1(0) = 2 \\ \dot{x}_2(t) = -\beta_1\ x_2(t) + \beta_2\ x_1(t)x_2(t)\ , & x_2(0) = 2 \end{array}\right\} \quad (19)$$

with

$$\alpha_1 = \alpha_2 = -t,\ \beta_1 = \beta_2 = t.$$

The exact solutions are given:

$$x_1(t) = \frac{2}{2-e^{\frac{t^2}{2}}},\ x_2(t) = \frac{2}{2-e^{\frac{t^2}{2}}}$$

By taking Shehu transformation on both sides of (19), we get

$$\left.\begin{array}{l} X_1(u,s) = \frac{2u}{s} + \frac{u}{s}S(\alpha_1\, x_1(t) - \alpha_2\, x_1(t)x_2(t)) \\ X_2(u,s) = \frac{2u}{s} + \frac{u}{s}S(-\beta_1\ x_2(t) + \beta_2\ x_1(t)x_2(t)) \end{array}\right\} \quad (20)$$

Taking the inverse Shehu transformation on both side of (20), so, we get,

$$\left.\begin{array}{l} x_1(t) = 2 + S^{-1}(\frac{u}{s}S(\alpha_1\ x_1(t) - \alpha_2\ x_1(t)x_2(t))) \\ x_2(t) = 2 + S^{-1}(\frac{u}{s}S(-\beta_1\ x_2(t) + \beta_2\ x_1(t)x_2(t))) \end{array}\right\} \quad (21)$$

By AGM , we must substitute (17) into (21), so, we get

$$\left.\begin{array}{l}\sum_{i=0}^{n} a_i\, t^i = 2 + S^{-1}\left(\frac{u}{s} S\left(\alpha_1 \sum_{i=0}^{n} a_i\, t^i - \alpha_2 \sum_{i=0}^{n} a_i\, t^i * \sum_{i=0}^{n} b_i\, t^i\right)\right) \\ \sum_{i=0}^{n} b_i\, t^i = 2 + S^{-1}\left(\frac{u}{s} S\left(-\beta_1 \sum_{i=0}^{n} b_i\, t^i + \beta_2 \sum_{i=0}^{n} a_i\, t^i * \sum_{i=0}^{n} b_i\, t^i\right)\right)\end{array}\right\} \quad (22)$$

When n =4, after simplification and offset values of $\alpha_1, \alpha_2, \beta_1$ *and* β_2, Eq. (22) becomes:

$$f(x_1(t)) = -t^9 a_4 b_4 + (-a_3 b_4 - a_4 b_3)t^8 + (-a_2 b_4 - a_3 b_3 - a_4 b_2)t^7 + (-a_1 b_4 - a_2 b_3 - a_3 b_2 - a_4 b_1)t^6 + (-b_3 a_1 - a_2 b_2 - a_3 b_1 + (-b_0 + 1)a_4 - b_4 a_0)t^5 + (-a_1 b_2 - a_2 b_1 + (-b_0 + 1)a_3 - b_3 a_0)t^4 + (-a_1 b_1 + (-b_0 + 1)a_2 - b_2 a_0 + 4a_4)t^3 + ((-b_0 + 1)a_1 - a_0 b_1 + 3a_3)t^2 + (2a_2 + (-b_0 + 1)a_0)t + a_1 = 0 \quad (23)$$

$$g(x_2(t)) = -t^9 a_4 b_4 + (-a_3 b_4 - a_4 b_3)t^8 + (-a_2 b_4 - a_3 b_3 - a_4 b_2)t^7 + (-a_1 b_4 - a_2 b_3 - a_3 b_2 - a_4 b_1)t^6 + (-a_3 b_1 - a_2 b_2 - b_3 a_1 + (-a_0 + 1)b_4 - b_0 a_4)t^5 + (-b_1 a_2 - a_1 b_2 + (-a_0 + 1)b_3 - b_0 a_3)t^4 + (-a_1 b_1 + (-a_0 + 1)b_2 - b_0 a_2 + 4b_4)t^3 + ((-a_0 + 1)b_1 - a_1 b_0 + 3b_3)t^2 + (2b_2 + (-a_0 + 1)b_0)t + b_1 = 0 \quad (24)$$

The constant coefficients of Eqs. (23) and (24) which are $\{a_0, \dots, a_4 \text{ and } b_0, \dots, b_4\}$ can be computed by applying boundary conditions in the form of:

$$x_1(t = 0) = 2 \rightarrow a_0 = 2, \quad x_2(t = 0) = 2 \rightarrow b_0 = 2. \quad (25)$$

On both Eqs. (23) and (24) and its derivatives, the initial conditions are applied as follows:

$$\begin{array}{l} f(x_1(t = 0)): a_1 = 0,\ g(x_2(t = 0)): b_1 = 0 \\ f'^{(x_1(t=0))}:\ -a_0\, b_0 + a_0 + 2a_2 = 0, \\ g'^{(x_2(t=0))}:\ -a_0\, b_0 + b_0 + 2b_2 = 0, \\ f''^{(x_1(t=0))}: -2a_0\, b_1 - 2a_1\, b_0 + 2a_1 + 6a_3 = 0, \\ g''^{(x_1(t=0))}: -2a_0\, b_1 - 2a_1\, b_0 + 2b_1 + 6b_3 = 0, \\ f'''^{(x_1(t=0))}: -6a_0\, b_2 - 6a_1\, b_1 - 6a_2\, b_0 + 6a_2 + 24a_4 = 0, \\ g'''^{(x_1(t=0))}: -6a_0\, b_2 - 6a_1\, b_1 - 6a_2\, b_0 + 6b_2 + 24b_4 = 0. \end{array}$$

With the help of Maple software, unknown constant coefficients a_i and b_i can now be found by solving the above equations, these coefficients are illustrated below:

$$a_1 = b_1 = 0, a_2 = b_2 = 1, a_3 = b_3 = 0 \text{ and } a_4 = b_4 = 3/4$$

Then, $x_1(t) = x_2(t) = 2 + t^2 + 0.750000t^4$ (26)

Now, take the Padé approximant of Eq. (26) ,with l = 2, m= 2, we get the solutions of the system (19):

$$x_{1[2,2]}(t) = x_{2[2,2]}(t) = \frac{2 - 0.5t^2}{1 - 0.75t^2}.$$

3.2. Example 2:

The second model takes into account the problem that the predator in the model is not significant from a commercial standpoint. The prey is continuously subjected to effort harvesting, which has no direct impact on the predator population. The availability of prey to the predator indirectly lowers the predator population. Furthermore, it is expected that the population of prey would rise simply logistically [38]. The following system represents this model:

$$\left.\begin{array}{l}\dot{x_1}\,(t) = x_1(t)\big(1 - x_1(t)\big) - bz(t) - rx_1(t) \\ \dot{x_2}\,(t) = c\,z(t) - e\,x_2(t)\end{array}\right\} \quad (27)$$

where

$$z(t) = \frac{x_1(t)\,x_2(t)}{x_1(t) + x_2(t)}.$$

For the numerical simulations of the model (27), we take

$$x_1(0) = 0.5\,, x_2(0) = 0.3, b = 0.8, c = 0.2, e = 0.5, r = 0.9.$$

By taking Shehu transformation on both sides of (27), we get

$$\left.\begin{array}{l}X_1(u\,,s\,) = \frac{0.5\,u}{s} + \frac{u}{s}\,S(x_1(t)\big(1 - x_1(t)\big) - bz(t) - rx_1(t)) \\ X_2(u\,,s\,) = \frac{0.3\,u}{s} + \frac{u}{s}\,S(c\,z(t) - e\,x_2(t))\end{array}\right\} \quad (28)$$

Taking the inverse Shehu transformation on both sides of (28), so, we get,

$$\left.\begin{array}{l}\mathrm{x}_1(\mathrm{t}) = 0.5 + \mathrm{S}^{-1}(\frac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}(x_1(t)\big(1 - x_1(t)\big) - bz(t) - rx_1(t))) \\ \mathrm{x}_2(\mathrm{t}) = 0.3 + \mathrm{S}^{-1}(\frac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}(c\,z(t) - e\,x_2(t)))\end{array}\right\} \quad (29)$$

By AGM, we must substitute (17) into (29), so, we get

$$\left.\begin{array}{l}\sum_{i=0}^{n} a_i\,t^i = 0.5 + \mathrm{S}^{-1}(\frac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}(\sum_{i=0}^{n} a_i\,t^i\big(1 - \sum_{i=0}^{n} a_i\,t^i\big) - \mathrm{bZ(t)} - \mathrm{r}\sum_{i=0}^{n} a_i\,t^i)) \\ \sum_{i=0}^{n} b_i\,t^i = 0.3 + \mathrm{S}^{-1}(\frac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}(\,\mathrm{cZ(t)} - \mathrm{e}\sum_{i=0}^{n} b_i\,t^i))\end{array}\right\} \quad (30)$$

where

$$Z(t) = \frac{\sum_{i=0}^{n} a_i\,t^i * \sum_{i=0}^{n} b_i\,t^i}{\sum_{i=0}^{n} a_i\,t^i + \sum_{i=0}^{n} b_i\,t^i}.$$

When n =3, after simplification and offset values of b, c, e and r, the Eq. (30) becomes:

$$f\big(x_1(t)\big) = 3t^2a_3 + 2ta_2 + a_1 - (t^3a_3 + t^2a_2 + ta_1 + a_0)(-t^3a_3 - t^2a_2 - ta_1 - a_0 + 1) + 0.8\,\mu + 0.9(t^3a_3 + t^2a_2 + ta_1 + a_0) = 0, \quad (31)$$

$$g\big(x_2(t)\big) = 3t^2b_3 + 2tb_2 + b_1 - 0.2\,\mu + 0.5(t^3b_3 + t^2b_2 + tb_1 + b_0) = 0\,, \quad (32)$$

Where

$$\mu = \frac{(t^3a_3 + t^2a_2 + ta_1 + a_0)(t^3b_3 + t^2b_2 + tb_1 + b_0)}{t^3(a_3 + b_3) + t^2(a_2 + b_2) + t(a_1 + b_1) + a_0b_0}$$

The constant coefficients of (31,32) which are $\{a_0\,, \dots, a_3$ and $b_0\,, \dots, b_3\,\}$ can be computed by applying boundary conditions in the form of:

$$x_1(t = 0) = 0.5 \rightarrow a_0 = 0.5, \qquad x_2(t = 0) = 0.3 \rightarrow b_0 = 0.3 \quad (33)$$

On both Eqs. (31) and (32), and its derivatives, the initial conditions are applied as follows:

$$f\big(\,x_1(t = 0)\big): a_1 - a_0(1 - a_0) + \frac{0.8a_0b_0}{a_0 + b_0} + 0.9a_0 = 0$$

$$g\big(x_2(t=0)\big): b_1 - \frac{0.2a_0b_0}{a_0+b_0} + 0.5\, b_0 = 0$$

$$f'^{\left(x_1(t=0)\right)}: 2a_2 - a_1(1-a_0) + a_0a_1 + 0.8\Big(\frac{a_1b_0}{a_0+b_0} + \frac{a_0b_1}{a_0+b_0} - \frac{a_0b_0(a_1+b_1)}{(a_0+b_0)^2}\Big) + 0.9a_1 = 0$$

$$g'^{\left(x_2(t=0)\right)}: 2b_2 - 0.2\Big(\frac{a_1b_0}{a_0+b_0} - \frac{a_0b_1}{a_0+b_0} + \frac{a_0b_0(a_1+b_1)}{(a_0+b_0)^2}\Big) + 0.5b_1 = 0$$

$$\begin{aligned} f''^{\left(x_1(t=0)\right)}&: 6a_3 - 2a_2(1-a_0) + 2a_1^2 + 2a_0a_2 \\ &+ 0.6\left(\frac{a_2b_0}{a_0+b_0} + \frac{a_1b_1}{a_0+b_0} - \frac{a_1b_0(a_1b_1)}{a_0+b_0} + \frac{a_0b_2}{a_0+b_0} - \frac{a_0b_1(a_1+b_1)}{(a_0+b_0)^2} + \frac{a_0b_0(a_1+b_1)^2}{(a_0+b_0)^3}\right) \\ &- \frac{0.8a_0b_0(2a_0+2b_2)}{(a_0+b_0)^2} + 1.8a_2 = 0 \end{aligned}$$

$$\begin{aligned} g''^{\left(x_1(t=0)\right)}&: 6b_3 \\ &- 0.4\left(\frac{a_2b_0}{a_0+b_0} - \frac{a_1b_1}{a_0+b_0} + \frac{a_1b_0(a_1+b_1)}{(a_0+b_0)^2} - \frac{a_0b_2}{a_0+b_0} + \frac{a_0b_1(a_1+b_1)}{(a_0+b_0)^2} - \frac{a_0b_0(a_1+b_1)^2}{(a_0+b_0)^3}\right) \\ &+ 0.2\frac{a_0b_0(2a_1+2b_1)}{(a_0+b_0)^2} + 1.0b_2 = 0 \end{aligned}$$

With the help of Maple software, unknown constant coefficients a_i and b_i can now be found by solving equations, these coefficients are given below:

$a_1 = -0.35000, a_2 = 0.19476562, a_3 = -0.10728816, b_1 = -0.112500, b_2 = 0.01880859, and\ b_3 = -0.00112847$

Then

$$\left.\begin{aligned} x_1(t) &= 0.5 = -0.35000t + 0.19476562t^2 - 0.10728816t^3 \\ \mathrm{x}_2(\mathrm{t}) &= 0.3 - 0.112500t + 0.01880859t^2 - 0.00112847t^3 \end{aligned}\right\} \qquad (34)$$

Now, take the Padé approximant of Eq. (34) ,with l = 2, m= 1, we get the solutions of the system (27):

$$x_{1_{[2,1]}}(t) = \frac{0.5 - 0.07457109077t + 0.001965388502t^2}{1 + 0.5508578185t}$$

$$x_{2_{[2,1]}}(t) = \frac{0.3 - 0.09450058413t + 0.01205881279t^2}{1 + 0.05999805297t}$$

3.3. Example 3:

The final model under consideration is the non-autonomous system of ordinary differential equations called a Lotka-Volterra model. In this model, time-varying values for the prey's growth rate, the predator's effectiveness in catching prey, the predator's mortality rate, and the predator's rate of growth are taken into account. It is significant to note that careful consideration must be given because this problem's coefficients are time-varying to acquire the proper recurrence equation system for the model. In other works [37, 39], the ordinary differential equation system shown below describes the aforementioned model:

$$\left.\begin{aligned} \dot{x_1}(t) &= \alpha_1\ x_1(t) - \alpha_2\ x_1(t)x_2(t) \\ \dot{x_2}(t) &= -\beta_1\ x_2(t) + \beta_2\ x_1(t)x_2(t) \end{aligned}\right\} \qquad (35)$$

For the numerical simulations of the model (35), we take

$\alpha_1 = 4 + tan(t)\,, \alpha_2 = exp(2t)\,, \beta_1 = -2, \beta_2 = cos(t)\,, x_1\,(0) = -4\ and$
$x_2(0) = 4$. The exact solution for these coefficients is

$$x_1(t) = \frac{-4}{cos(t)}\quad , \qquad x_2(t) = 4\, exp(-2t)\,.$$

By taking Shehu transformation on both sides of (35), we get

$$\left.\begin{aligned} X_1(u\,,s\,) &= -\tfrac{4\,u}{s} + \tfrac{u}{s}\, S(\,\alpha_1\, x_1(t) - \,\alpha_2\, x_1(t)x_2(t)) \\ X_2(u\,,s\,) &= \tfrac{4\,u}{s} + \tfrac{u}{s}\, S(-\,\beta_1\,\ x_2(t) + \,\beta_2\,\ x_1(t)x_2(t)) \end{aligned}\right\} \tag{36}$$

Taking the inverse Shehu transformation on both sides of (36), we get,

$$\left.\begin{aligned} \mathrm{x}_1(\mathrm{t}) &= -4 + \mathrm{S}^{-1}(\,\tfrac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}(\,\alpha_1\,\ \mathrm{x}_1(\mathrm{t}) - \,\alpha_2\,\ \mathrm{x}_1(\mathrm{t})\mathrm{x}_2(\mathrm{t})\,)) \\ \mathrm{x}_2(\mathrm{t}) &= 4 + \mathrm{S}^{-1}(\,\tfrac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}(-\,\beta_1\,\ \mathrm{x}_2(\mathrm{t}) + \,\beta_2\,\ \mathrm{x}_1(\mathrm{t})\mathrm{x}_2(\mathrm{t})\,)) \end{aligned}\right\} \tag{37}$$

By AGM , we must substitute (17) in (37), so, we get

$$\left.\begin{aligned} \textstyle\sum_{i=0}^{n} a_i\, t^i &= -4 + \mathrm{S}^{-1}(\,\tfrac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}\big(\,\alpha_1 \textstyle\sum_{i=0}^{n} a_i\, t^i - \,\alpha_2 \sum_{i=0}^{n} a_i\, t^i * \sum_{i=0}^{n} b_i\, t^i\,)\big) \\ \textstyle\sum_{i=0}^{n} b_i\, t^i &= 4 + \mathrm{S}^{-1}(\,\tfrac{\mathrm{u}}{\mathrm{s}}\,\mathrm{S}\big(-\,\beta_1 \textstyle\sum_{i=0}^{n} b_i\, t^i + \,\beta_2 \sum_{i=0}^{n} a_i\, t^i * \sum_{i=0}^{n} b_i\, t^i\,)\big) \end{aligned}\right\} \tag{38}$$

When n =2, after simplification and offset values of $\alpha_1, \alpha_2, \beta_1\ and\ \beta_2$, Eq. (38) becomes:

$$f\big(x_1(t)\big) = 2ta_2 + a_1 - (4 + \tan(t))(t^2a_2 + ta_1 + a_0) + e^{2t}(t^2a_2 + a_1 + a_0)(t^2b_2 + tb_1 + b_0) = 0 \tag{39}$$

$$g\big(x_2(t)\big) = 2tb_2 + b_1 - 2t^2b_2 - 2tb_1 - 2b_0 - \cos(t)\,(t^2a_2 + ta_1 + a_0)(t^2b_2 + tb_1 + b_0) = 0 \tag{40}$$

The constant coefficients of (39,40) which are $\{a_0\,, a_1, a_2\ and\ b_0\,, b_1, b_2\}$ can be computed by applying boundary conditions in the form:

$x_1(t = 0) = -4 \rightarrow a_0\ = -4, \qquad x_2(t = 0) = 4 \rightarrow b_0\ = 4.$

On both Eqs. (39,40) and its derivatives, the initial conditions are applied as follows:

$$\begin{aligned} &f\big(\,x_1(t = 0)\big){:}\ a_0\, b_0 - 4a_0 + a_1 = 0, \\ &g\big(\,x_2(t = 0)\big){:} - a_0b_0 - 2b_0 + b_1 = 0, \\ &f'^{\left(\,x_1(t=0)\right)} : 2a_0b_0 + a_0b_1 + a_1b_0 - a_0 - 4a_1 + 2a_2\ = 0, \\ &g'^{\left(\,x_2(t=0)\right)} :\ -a_0b_1 - a_1b_0 - 2b_1 + 2b_2 = 0, \end{aligned}$$

With the help of Maple software, unknown constant coefficients a_i and b_i can now be found by solving equations, these coefficients are as below:

$a_0 = -4, a_1 = 0, a_2 = -2, b_0 = 4, b_1 = -8, b_2 = 8.$

Then,

$$\left.\begin{aligned} x_1(t) &= -4 - 2t^2 \\ x_2(t) &= 4 - 8t + 8t^2 \end{aligned}\right\} \tag{41}$$

Now, taking the Padé approximant of Eq. (41) ,with l = 0, m= 2, we get the solutions of the system (35):

$$x_{1\,[0,2]}(t) = \frac{-4}{1-0.5t^2}$$

$$x_{2\,[0,2]}(t) = \frac{4}{1+2t+2t^2}$$

4. Results and discussion

This section, discusses the numerical computations for predator- prey models, which have been obtained by the application of SAGPM. All calculations are employed by Maple 2016 software. In Tables1, 2 and 3, a comparison was made between the errors, the central processing unit, and the rate of convergence obtained using AGM, modified decomposition method (ADM) [38], He's variational iteration method (VIM) [15], Homotopy perturbation method (HPM) [37], Adomian decomposition method(ADM) [39] and SAGPM. It was noted through the results that the proposed method is characterized by high accuracy, fewer errors, and a lower central processing unit when compared to other methods. Fig.1 shows the comparison between the solution offered by the new method SAGPM, AGM, ADM[38], VIM [15] and exact solutions for $x_1(t)$ *and* $x_2(t)$ for Example1.

Fig.2 shows the comparison between the solution offered by the new method SAGPM, AGM, HPM [37] and ADM [39] for $x_1(t)$ *and* $x_2(t)$ respectively for Example2.

Fig.3 shows the comparison between the solution offered by the new method SAGPM, AGM, ADM [39] and exact solutions for $x_1(t)$ *and* $x_2(t)$ by using second terms for Example 3. According to the calculations demonstrated in the tables and figures, the SAGPM processes are particularly effective at solving prey-predator systems, and the SAGPM is the most efficient one. It also gives high-precision solutions since it yields good results with solution iterations and errors.

Table 1. Comparison of error, CPU time and rate of convergence between SAGPM, AGM,VIM, and ADM, when $t \in [0,1]$ for Example 1

Functions	*Errors*	*SAGPM*	*AGM*	*VIM [15]*	*ADM [38]*
$x_1(t)$=$x_2(t)$	L_2	0.06391072710	0.4716321433	0.3006935003	0.4716321433
	L_∞	0.306515496	1.94348450	1.343979181	1.943484504
	CPU(s)	0.015	0.031	0.031	0.015
	Rate	8.24	0.95	0.95	0.95

Table 2. Comparison of error, CPU time and rate of convergence between SAGPM, AGM,HPM and ADM, when $t \in [0,5]$ for Example 2.

Functions	*Errors*	*SAGPM*	*AGM*	*HPM [37]*	*ADM [39]*
$x_1(t)$	L_2	0.01099356035	12.33180023	12.33178100	12.33086147
	L_∞	0.00962745319	13.41102090	13.41100000	13.41000000
	CPU(s)	0.016	0.032	0.031	0.031
	Rate	2.78	1.01	1.01	1.01
$x_2(t)$	L_2	0.01797838041	0.1297084105	0.1296993297	0.1297223179
	L_∞	0.01226168904	0.1410598755	0.1410500000	0.1410750000
	CPU(s)	0.015	0.031	0.016	0.016
	Rate	1.22	0.55	0.55	0.55

Table 3. Comparison of error, CPU time and rate of convergence between SAGPM,AGM and ADM ,when $t \in [0,1]$ for Example 3.

Functions	*Errors*	*SAGPM*	*AGM*	*ADM [39]*
$x_1(t)$	L_2	0.02399586585	0.06355437342	0.06355437342
	L_∞	0.596737128	1.403262872	1.403262872
	CPU(s)	6.000	6.796	6.796
	Rate	4.175	1.00	------
$x_2(t)$	L_2	0.02226227582	0.2027234285	0.2027234285
	L_∞	0.2586588672	3.458658867	3.458658867
	CPU(s)	6.421	7.234	7.234
	Rate	2.321	0.68	------

Where; the measurement errors are defined as the following:

$$\|E\|_{L_2} = \sqrt{h \sum_{i=1}^{n} |x_i^{exact} - x_i^{approx}|^2} \ ,$$

$$\|E\|_{L_\infty} = max_{i=0..n}(|x_i^{exact} - x_i^{approx}|) \ ,$$

and the convergence rate is defined as the following:

$$Rate = \frac{\log\left(\frac{E_{i+1}}{E_i}\right)}{\log\left(\frac{E_i}{E_{i-1}}\right)}$$

Where E_{i-1}, E_i and E_{i+1} have represented the error in steps $i-1$, i, and $i+1$, respectively.

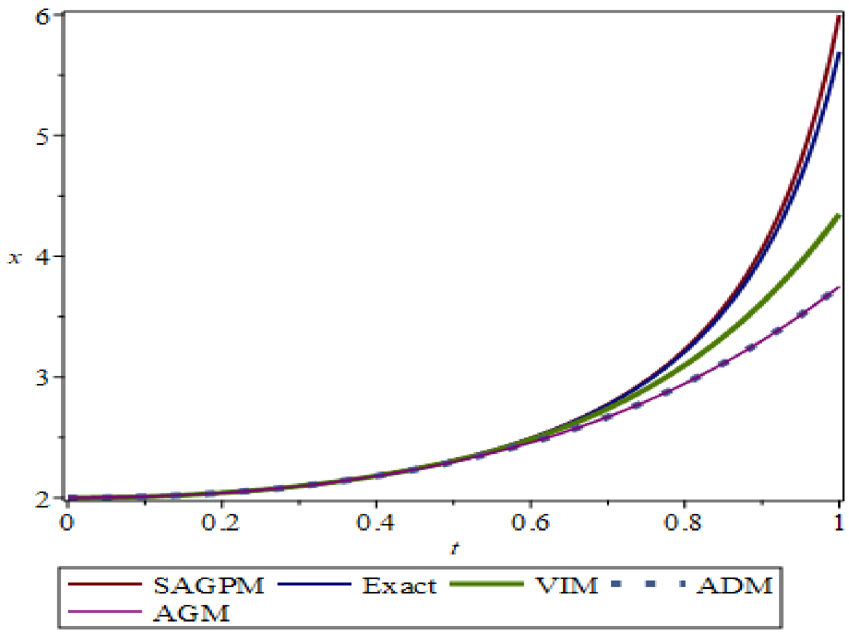

Fig 1. Comparison of approximate solutions obtained by SAGPM, VIM,ADM, and AGM with exact solution for Example 1, ($x_1(t) = x_2(t)$) .

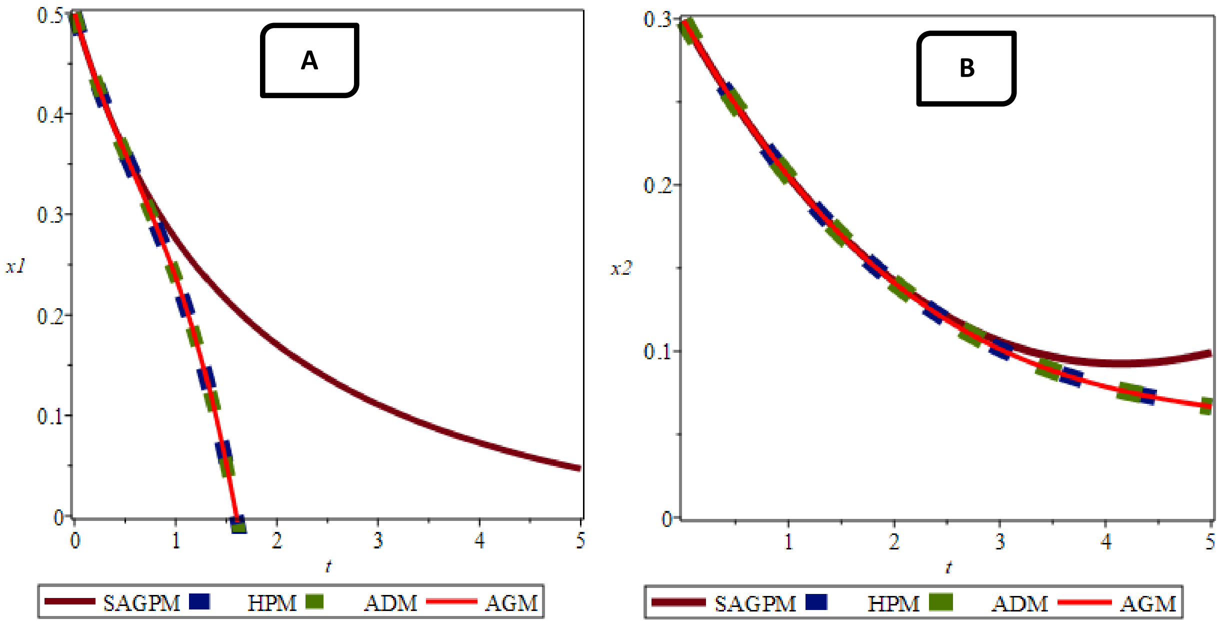

Fig. 2. Comparison of approximate solutions obtained by AGM,HPM,ADM and SAGPM for $x_1(t)$ in A and $x_2(t)$ in B for Example 2.

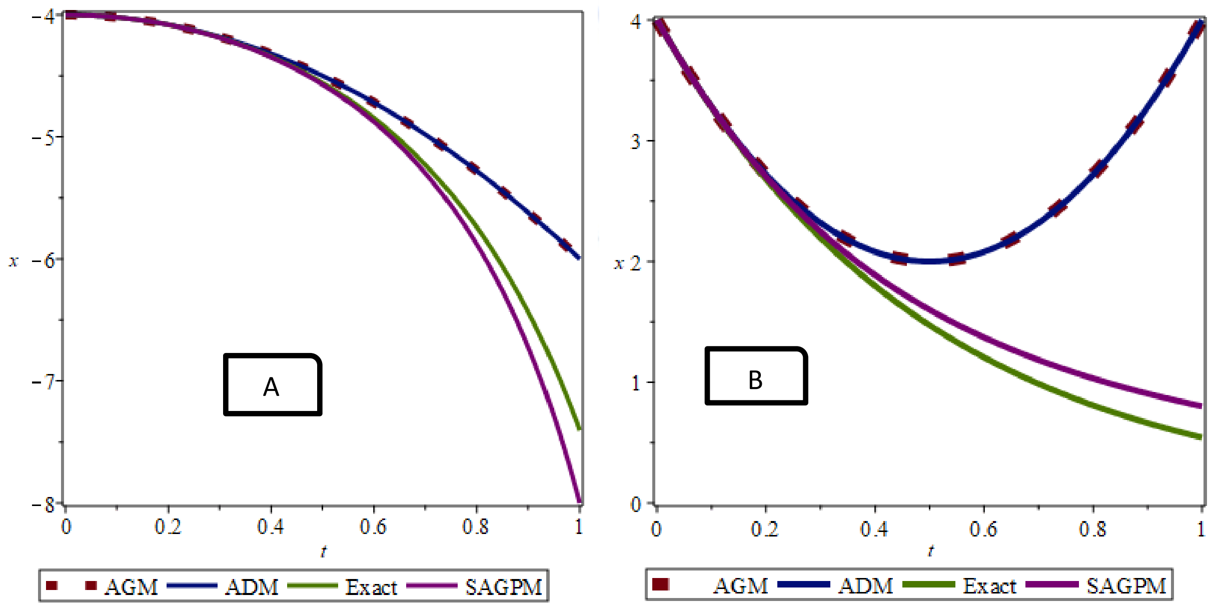

Fig. 3. Comparison of approximate solutions obtained by AGM,ADM and SAGPM with the exact solution for $x_1(t)$ in A and $x_2(t)$ in B for Example 3.

5. Convergence analysis of SAGPM

This part, illustrates how the approximate analytical solutions from SAGPM for systems (19), (27) and (35) converge.

Consider the system of equations in the following form:

$$x_1 = F(x_1, x_2), \qquad (42)$$

$$x_2 = G(x_1, x_2),$$

where F and G are non-linear operators. The solution by using the present approach is equivalent to the following sequence:

$$S_n = \sum_{j=0}^{n} x_{1j} = \sum_{j=0}^{n} a_j \frac{t^j}{(j)!}$$
$$K_n = \sum_{j=0}^{n} x_{2j} = \sum_{j=0}^{n} b_j \frac{t^j}{(j)!}$$

Theorem (5.1): (Convergence of Systems)[40]

Let H be a Hilbert space, and let F, G be an operator from H into H, and x_1, x_2 be the exact solution of Eq.(42). The approximate solutions

$$\sum_{j=0}^{\infty} x_{1j} = \sum_{j=0}^{\infty} a_j \frac{t^j}{(j)!}$$

and

$$\sum_{j=0}^{\infty} x_{2j} = \sum_{j=0}^{\infty} b_j \frac{t^j}{(j)!}$$

are convergence to exact solutionsx_1 , x_2 respectively when $0 \le \rho < 1$,

$\left\|x_{1_{j+1}}\right\| \le \rho \left\|x_{1_j}\right\|, \forall i \in N \cup \{0\}$ and $\exists 0 \le \sigma < 1, \left\|x_{2_{j+1}}\right\| \le \sigma \left\|x_{2_j}\right\|, \forall j \in N \cup \{0\}$.

Definition (5.2): For every n ∈ $N \cup \{0\}$,we define

$$\rho_j = \begin{cases} \frac{\left\|x_{1_{j+1}}\right\|}{\left\|x_{1_j}\right\|}, & \left\|x_{1_j}\right\| \ne 0 \\ 0 \;, & otherwise \end{cases} \quad \text{and} \quad \sigma_j = \begin{cases} \frac{\left\|x_{2_{j+1}}\right\|}{\left\|x_{2_j}\right\|}, & \left\|x_{2_j}\right\| \ne 0 \\ 0 \;, & otherwise \end{cases}$$

Corollary(5.3): From the theorem above $\sum_{j=0}^{\infty} x_{1j} = \sum_{j=0}^{\infty} a_j \frac{t^j}{(j)!}$ and $\sum_{j=0}^{\infty} x_{2j} = \sum_{j=0}^{\infty} b_j \frac{t^j}{(j)!}$ convergence to exact solutions x_1 and x_2 when $0 \le \rho_j < 1$ and $0 \le \sigma_j < 1, j = 0,1,2, \ldots$

Now, to demonstrate how the three cases' analytical approximations converge, we have applied the corollary as follows;

In the first example, at $t \in [0,1]$ for $x_1 = x_2$, we get

$$\rho_1 = \frac{\|x_{1_1}\|}{\|x_{1_0}\|} = 0.9999875001 < 1,$$
$$\rho_2 = \frac{\|x_{1_2}\|}{\|x_{1_1}\|} = 0.5000250018 < 1,$$
$$\rho_3 = \frac{\|x_{1_3}\|}{\|x_{1_2}\|} = 0.3535666492e - 4 < 1.$$

In the second example, at $t \in [0,5]$ for x_1, we get

$$\rho_1 = \frac{\|x_{1_1}\|}{\|x_{1_0}\|} = 0.9939414480 < 1,$$
$$\rho_2 = \frac{\|x_{1_2}\|}{\|x_{1_1}\|} = 0.2952666150e - 1 < 1,$$
$$\rho_3 = \frac{\|x_{1_3}\|}{\|x_{1_2}\|} = 0.5302789404e - 2 < 1.$$

and for x_2

$$\sigma_1 = \frac{\|x_{2_1}\|}{\|x_{2_0}\|} = 0.9939414480 < 1,$$
$$\sigma_2 = \frac{\|x_{2_2}\|}{\|x_{2_1}\|} = 0.3958688457e - 2 < 1,$$
$$\sigma_3 = \frac{\|x_{2_3}\|}{\|x_{2_2}\|} = 0.6461243933e - 3 < 1.$$

In the third example, at $t \in [0,1]$ for x_1, we get

$$\rho_1 = \frac{\|x_{1_1}\|}{\|x_{1_0}\|} = 0.2500052087 < 1,$$
$$\rho_2 = \frac{\|x_{1_2}\|}{\|x_{1_1}\|} = 0.1414184098e - 3 < 1,$$
$$\rho_3 = \frac{\|x_{1_3}\|}{\|x_{1_2}\|} = 0.4166666666e - 4 < 1.$$

and for x_2

$$\sigma_1 = \frac{\|x_{2_1}\|}{\|x_{2_0}\|} = 0.9950455635 < 1,$$
$$\sigma_2 = \frac{\|x_{2_2}\|}{\|x_{2_1}\|} = 0.7035684045e - 2 < 1,$$
$$\sigma_3 = \frac{\|x_{2_3}\|}{\|x_{2_2}\|} = 0.3333333338e - 2 < 1.$$

6. Stability analysis

The future state of a dynamical system can be described by its evolution, which is a fixed phenomenon. Stability is often the first and most crucial problem to be addressed when trying to determine how a system behaves. This section's goal is to look into the stability analysis of the SAGPM, which is used to solve the prey -predator systems. The analysis of the model is largely influenced by the equilibrium point of a system of nonlinear differential equations for the majority of dynamical systems. The eigenvalues of the Jacobian matrix of the associated dynamical system are calculated to categorize the equilibrium locations.

In Fig.4 (A) when both $x_1 > 0$ and $x_2 > 0$ then some trajectories are almost elliptical around the point (1,1). we begin with both the prey and predator population as very small. We have noticed from the figure that the prey first increases because there is little predation. After a while, the predator population increase because of a bigger food supply. This decreases the prey population as we have seen in the figure. With less abundance of prey, the system goes back to its initial form as the predator population starts to decline once more. Then the trajectory begins to repeat itself. In Fig.4 (B) We noted that the critical point is unstable. The solutions are unstable near point (0,0). If the critical point is stable, this leads to the attraction of non-zero groups to it, which leads to the extinction of both species. Therefore, the instability of this point is very important. However, the extinction of both species is challenging to represent because the fixed point at the origin is a saddle point, which is unstable. This is only possible if the whole population of prey is purposely wiped off, starving the predators to death. In this straightforward model, if the predators disappeared, the number of prey would increase unchecked. Fig.4 (C) shows some solutions approach the critical point (0,0) as t→ 0. The critical point is stable and we naturally also see that the trajectory which corresponds to our initial condition also approaches the critical point. That is, the solutions are stable near this point. While the solutions are unstable near point (-2,3.5) because it is unstable.

7. Conclusion

In order to solve the predator-prey systems, a new analytical approximation method was created and used in this study. In comparison to the existing approaches like ADM, VIM, HPM, and AGM, the numerical results from the three cases show that the new technique offers good accuracy, especially for large values of time t. We have observed from the results that the new method is a mathematical tool for solving predator-prey models with constant or variable coefficients. Also, it is a crucial technique for solving beginning and boundary value problems that come up in a variety of applied sciences disciplines, such as mechanics, physics, applied mathematics, and others. It has been noted through tables and graphs that the proposed method noticeable convergence and high accuracy compared to other methods. In addition, the new technique is an improvement of Akbari- Ganji's method. We will focus our future work on researching a few fractional- order predator-prey systems.

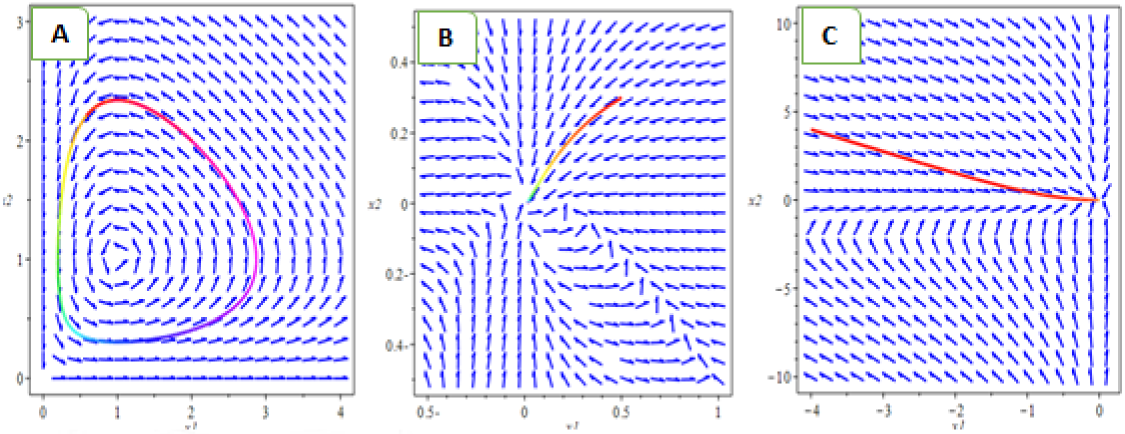

Fig.4: Direction fields and trajectory to systems (19), (27), and (35) at $t = 20$ and using the parameter values $\alpha_1 = \alpha_2 = 0.5$ and $\beta_1 = \beta_2 = 0.3$ in (A), $b = 0.8, c = 0.2, e = 0.5, r = 0.9$ in (B), $\alpha_1 = -4, \alpha_2 = 2, \beta_1 = 7, \beta_2 = 2$ in (C).

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