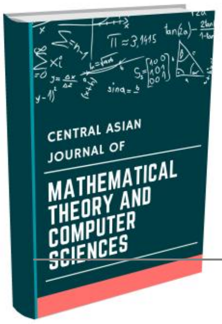

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COMPARATIVE ANALYSIS OF LAPLACE VARIATIONAL ITERATION AND LAPLACE HOMOTOPY PERMUTATION METHODS FOR APPROXIMATE SOLUTIONS WITH ATANGANA-BALEANU OPERATOR

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Abstract

This research paper introduces and compares the efficacy of two techniques, namely the Laplace Variational Iteration Method (LVIM) and the Laplace Homotopy Permutation Method (LHPM), in obtaining approximate solutions for fractional differential equations (FDEs) incorporating the Atangana-Baleanu fractional derivative in the Caputo sense (ABCO). The Atangana-Baleanu fractional derivative, a specialized form of fractional calculus, offers a more accurate representation of anomalous diffusion processes.

The LVIM and LHPM are employed to tackle fractional differential equations, which frequently arise in diverse scientific and engineering disciplines. The LVIM combines the advantages of the Laplace transform and the variational iteration technique to yield a reliable and efficient solution strategy. On the other hand, the LHPM integrates the Laplace transform and the homotopy permutation technique to address the same class of equations.

This paper provides a detailed exposition of both methodologies, including their algorithmic frameworks and computational procedures. The comparison of the two techniques involves assessing their performance in approximating solutions to fractional differential equations with the Atangana-Baleanu fractional derivative. The obtained results underscore the success of both methods in yielding accurate and practical approximate solutions for this specific type of fractional differential equation.

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Introduction

Fractional differential equations (FDEs) have emerged as a versatile mathematical tool to model a wide range of complex phenomena in various scientific and engineering domains. These equations involve fractional derivatives, which capture the effects of memory and long-range interactions in dynamical systems. The utilization of fractional calculus in problem-solving has been shown to provide more accurate representations of anomalous diffusion, viscoelasticity, and other intricate processes that cannot be adequately described by classical integer-order derivatives. Among the various types of fractional derivatives, the Atangana-Baleanu fractional derivative in the Caputo sense has garnered considerable attention due to its ability to address the shortcomings of traditional fractional derivatives in accurately capturing non-local effects. As such, the development of effective solution methods for FDEs incorporating the Atangana-Baleanu fractional derivative is of paramount importance in advancing the understanding of real-world phenomena[1], [2].

In this context, this research paper introduces and explores two innovative solution techniques: the Laplace Variational Iteration Method (LVIM) and the Laplace Homotopy Permutation Method (LHPM). These methods are designed to provide approximate solutions for FDEs featuring the Atangana-Baleanu fractional derivative. The LVIM combines the power of the Laplace transform and the variational iteration technique, while the LHPM integrates the Laplace transform with the homotopy permutation approach. The primary objective of this study is to present a comprehensive comparison of the LVIM and LHPM in terms of their capabilities to yield accurate and practical approximate solutions for FDEs incorporating the Atangana-Baleanu fractional derivative. By analyzing their algorithmic frameworks, computational procedures, and performance metrics, this paper aims to shed light on the strengths and limitations of these methods in tackling this specific class of fractional differential equations[3], [4].

Through this comparative analysis, researchers and practitioners working on problems involving the Atangana-Baleanu fractional derivative can gain insights into selecting suitable solution techniques for their applications. Moreover, this research contributes to the broader field of fractional calculus by evaluating the efficacy of these emerging methods and highlighting avenues for further refinement and advancement. The subsequent sections of this paper provide a detailed exposition of the LVIM and LHPM, followed by a presentation of their computational results when applied to representative FDEs with the Atangana-Baleanu fractional derivative. The findings of this study underscore the successful applicability of both LVIM and LHPM, fostering a deeper understanding of their potential contributions to solving complex fractional differential equations in diverse scientific and engineering contexts[5]–[11].

In this paper, we apply LVIM, and LHPM to find solution of fractional differential equations with the fractional operator Atangana-Baleanu. The order of the paper is as follows: The basic definitions for calculus and fractional integration are presented in section 2, the methods used are analyzed in section 3, many examples are given that explain the effectiveness of the method proposed in section 4, and finally, the conclusion is provided in section 5.

Basic concepts

Definition 1 [12], [13]The Atangana-Baleanu-Caputo operator when $f \in H^1(a,b), a > b$ and for $0 < \alpha \leq 1$ is,

$$^{ABC}D_\tau^\alpha f(\tau) = \frac{B(\alpha)}{1-\alpha} \int_0^\tau f'(x)\, E_\alpha\left(-\frac{\alpha}{1-\alpha}(\tau - x)^\alpha\right) dx, \quad (1)$$

where $B(\alpha)$ is a function such that $B(0) = B(1) = 1$.

Definition 2 [14]Assume set of function,

$$A = \left\{ f(\tau) \middle| \exists M, z_1, z_2 > 0, |f(\tau)| < M e^{\frac{|\tau|}{z_j}}, \tau \in (-1)^j \times [0, \infty) \right\},$$

Then the Laplace transform over A set

$$L\big(f(\tau)\big) = F(s) = \int_0^\infty e^{-s\tau} f(\tau)\, dt. (2)$$

The Laplace transform of The Atangana-Baleanu-Caputo operator is given by [13]

$$L\left({}^{ABC}D_\tau^\alpha f(\tau)\right) = \frac{B(\alpha)}{1-\alpha+\alpha s^\alpha}\left(F(s) - \frac{1}{s}f(0)\right). (3)$$

Definition 3 [15], [16] The inverse Laplace transform of a function is defined by

$$L^{-1}(F(s)) = f(\tau) = \frac{1}{2i\pi}\int_{p-i\infty}^{p+i\infty} e^{s\tau} F(s)d\tau, (5)$$

where s is Laplace transform variable and p is a real constant.

Analysis of LVIM

Let's considering FDEs with ABCO

$${}^{ABC}D_\tau^\alpha u(\mu,\tau) + N\big(u(\mu,\tau)\big) = f(\mu,\tau), (6)$$

with initial condition $u(\mu,0) = u_0(\mu)$, where ${}^{ABC}D_\tau^\alpha$ is the FABCO, N is the nonlinear operator and $f(\mu,\tau)$ is a source term.

We obtain by applying the Laplace transform to Eq.(6) with the stated initial condition

$$\frac{B(\alpha)}{1-\alpha+\alpha s^\alpha}\left(L(u) - \frac{1}{s}u_0\right) = L[f(\mu,\tau) - N(u)], (7)$$

by substituting initial condition of Eq.(8),

$$\bar{u} = \frac{1}{s}u_0(\mu) - \frac{1-\alpha+\alpha s^\alpha}{B(\alpha)}L[f(\mu,\tau) - N(u)], (8)$$

appling variation intretion method

$$\bar{u}_{n+1} = \bar{u}_n + \lambda\left(\bar{u}_n - \frac{1}{s}u_0(\mu) + \frac{1-\alpha+\alpha s^\alpha}{B(\alpha)}L[N(u_n) - f(\mu,\tau)]\right), (9)$$

Note that λ is the Lagrange multiplier, and since $0 < \alpha < 1$, hence $\lambda = -1$, we obtain after applying the inverse of the Laplace transform to both sides of the equation

$$u_{n+1} = u_0(\mu) - L^{-1}\left(\frac{1-\alpha+\alpha s^\alpha}{B(\alpha)}L[N(u_n) - f(\mu,\tau)]\right), (10)$$

where we have the initial iteration is $u(\mu,0) = u_0(\mu)$, hence

$$u(\mu,\tau) = \lim_{k\to\infty} u_k(\mu,\tau).$$

Analysis of LHPM

Subject to the provided initial condition, apply the Laplace transform to Eq.(6).

$$\frac{B(\alpha)}{1-\alpha+\alpha s^\alpha}\left(L(s) - \frac{1}{s}u_0\right) = L(g(x,t) - \mathrm{N}\,[u(x,t)]), (11)$$

by substituting initial condition of Eq.(11),

$$\bar{u} = \frac{1}{s}u_0(x) - \frac{1-\alpha+\alpha s^{\alpha}}{B(\alpha)}L(\mathrm{N}\,[u] - g), (12)$$

Taking the inverse of the Laplace transform and applying it to both sides of the equation (12),

$$u = u_0(x) + L^{-1}\left(\frac{1-\alpha+\alpha s^{\alpha}}{B(\alpha)}L(g)\right) - L^{-1}\left(\frac{1-\alpha+\alpha s^{\alpha}}{B(\alpha)}L(\mathrm{N}\,[u])\right), (13)$$

applying homotopy permutation method,

$$u(x,t) = \sum_{\mathrm{n}=0}^{\infty} p^n u_n(x,t)\,, \mathrm{N}[u(x,t)] = \sum_{\mathrm{n}=0}^{\infty} p^{\mathrm{n}} \mathcal{H}_{\mathrm{n}}(u)$$

where

$$\mathcal{H}_{\mathrm{n}}(u_1, u_2, u_3, \dots, u_n) = \frac{1}{\mathrm{n}!}\frac{\partial^{\mathrm{n}}}{\partial p^{\mathrm{n}}}[\mathrm{N}(\sum_{\mathrm{i}=0}^{\mathrm{n}} p^i u_i(x,t))]_{p=0}$$

Substituting Eq.(14) into Eq.(13) gives us the result that,

$$\sum_{n=0}^{\infty} p^n u_n\,(x,t) = \mathcal{G}(x,t) - p\left(L^{-1}\left(\frac{1-\alpha+\alpha s^{\alpha}}{B(\alpha)}L\left(\sum_{\mathrm{n}=0}^{\infty} p^{\mathrm{n}}\mathcal{H}_{\mathrm{n}}(u)\right)\right)\right), (15)$$

on comparing both sides of Eq.(15), the following result is obtained,

$$p^0 : u_0(x,t) = \mathcal{G}(x,t),$$

$$p^1 : u_1(x,t) = -L^{-1}\left(\frac{1-\alpha+\alpha s^{\alpha}}{B(\alpha)}L\big(\mathcal{H}_0(u)\big)\right),$$

$$\vdots$$

$$p^n : u_n(x,t) = -L^{-1}\left(\frac{1-\alpha+\alpha s^{\alpha}}{B(\alpha)}L\big(\mathcal{H}_{\mathrm{n}-1}(u)\big)\right), (16)$$

using the parameter p, we expand the solution in the following form

$$u(x,t) = \sum_{\mathrm{n}=0}^{\infty} p^n u_n(x,t)\,, (17)$$

choose $p = 1$ results in the solution of Eq.(11)

$$u(x,t) = \lim_{p\to 1}\sum_{\mathrm{n}=0}^{\infty} p^n u_n(x,t) = \sum_{\mathrm{n}=0}^{\infty} u_n(x,t)\,.\,(18)$$

Application

In this part, two nonlinear equations and a linear system will be solved using the LVIM and LHPM techniques, with the assumption that $\boldsymbol{B(\alpha) = 1}$.

Example 1 Consider the nonlinear equation with the ABCO

$${}^{\mathcal{ABC}}\mathcal{D}_{\tau}^{\alpha} u(\mu,\tau) = -\frac{\partial}{\partial\mu}\left(\frac{12}{\mu}u - \mu\right)u + \frac{\partial^2}{\partial\mu^2}u^2, 0 < \alpha \le 1, (19)$$

depending on the initial condition $u(\mu,0) = \mu^2$.

Below we present the LVIM,

applying the LVIM to Eq.(19), we get

$$u_{n+1} = \mu^2 - L^{-1}\left((1-\alpha+\alpha s^\alpha)L\left(\frac{12}{\mu}u_{\mu n}u_n - \frac{12}{\mu^2}u_n^2 - (u_n^2)_{\mu\mu} + u_n\right)\right), (20)$$

now, we find the approximate solutions as,

$$u_0 = \mu^2,$$

$$u_1 = \mu^2 + L^{-1}\big((1-\alpha+\alpha s^\alpha)L(\mu^2)\big)$$

$$= \mu^2\left((2-\alpha) + \alpha\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right),$$

$$u_2 = \mu^2 + L^{-1}\left((1-\alpha+\alpha s^\alpha)L\left((2-\alpha)+\alpha\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right)\right)$$

$$= \mu^2\left(1 + (1-\alpha)(2-\alpha) + (\alpha(1-\alpha)+\alpha(2-\alpha))\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \alpha^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right), (21)$$

and so on.

Therefore, the series solution $u(\mu,\tau)$ of Eq.(19) is given by

$$u(\mu,\tau) = \mu^2\left((\alpha^2-3\alpha+3) + (3\alpha-2\alpha^2)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \alpha^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right), (22)$$

when choosing $\alpha = 1$ in Eq.(22), it becomes

$$u(\mu,\tau) = \mu^2\left(1 + \tau + \frac{\tau^2}{2!} + \cdots\right), (23)$$

Now, we present the LHPM,

applying the LHPM to Eq.(19), we get

$$\sum_{n=0}^{\infty}\wp^n u_n = \mu^2 - \wp L^{-1}\left[(1-\alpha+\alpha s^\alpha)L\begin{bmatrix}\sum_{n=0}^{\infty}\wp^n\mathcal{A}_n - \sum_{n=0}^{\infty}\wp^n \mathrm{B}_n\\ +\sum_{n=0}^{\infty}\wp^n\mathcal{C}_n + \sum_{n=0}^{\infty}\wp^n u_n\end{bmatrix}\right], (24)$$

by comparing both sides of the Eq.(24), the following result is obtained,

$$\wp^0 : u_0 = \mu^2,$$

$$\wp^1 : u_1 = L^{-1}\big[(1-\alpha+\alpha s^\alpha)L[\wp^0\mathcal{A}_0 - \wp^0\mathrm{B}_0 + \wp^0\mathcal{C}_0 + \wp^0 u_0]\big],$$

$$\wp^2 : u_2 = L^{-1}\big[(1-\alpha+\alpha s^\alpha)L[\wp^1\mathcal{A}_1 - \wp^1\mathrm{B}_1 + \wp^1\mathcal{C}_1 + \wp^1 u_1]\big],$$

by the above algorithms,

$$u_0 = \mu^2,$$

$$u_1 = \mu^2\left(1 - \alpha + \alpha\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right),$$

$$u_2 = \mu^2\left((1-2\delta+\delta^2) + (2\delta-2\delta^2)\frac{\tau^\alpha}{\Gamma(\alpha+1)} + \delta^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right),$$

and so on.

Therefore, the series solution $u(\mu,\tau)$ of Eq. (19) is given by

$$u(\mu,\tau) = \mu^2\left[(3-3\alpha+\alpha^2)+(3\alpha-2\alpha^2)\frac{\tau^\alpha}{\Gamma(\alpha+1)}+\delta^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right].(25)$$

If we put $\alpha\to 1$ in Eq.(25) , we get the approximate and exact solution

$$u(\mu,\tau) = \mu^2\left(1+\frac{\tau}{1!}+\frac{\tau^2}{2!}+\cdots\right).(26)$$

Ultimately, the exact solution of Eq.(19),

$$u(\mu,\tau) = \mu^2 e^\tau.(27)$$

Tab.1 The values of Eqs.(22,23,25,26,27) at different values of $t, x, and\ \alpha$.

μ	τ	$u_{\alpha=0.8}$	$u_{\alpha=0.9}$	$u_{\alpha=1}$	u_{exact}	$\lvert u_e - u_1\rvert$
0.0417	0.0417	0.0023	0.0020	0.0018	0.0018	0.0000
0.1250	0.1250	0.0232	0.0202	0.0177	0.0177	0.0000
0.2083	0.2083	0.0703	0.0613	0.0534	0.0535	0.0001
0.2917	0.2917	0.1490	0.1304	0.1135	0.1139	0.0004
0.3750	0.3750	0.2646	0.2330	0.2032	0.2046	0.0014
0.4167	0.4167	0.3381	0.2987	0.2610	0.2634	0.0023
0.5000	0.5000	0.5196	0.4626	0.4062	0.4122	0.0059
0.5833	0.5833	0.7521	0.6753	0.5967	0.6098	0.0131
0.6667	0.6667	1.0415	0.9433	0.8395	0.8657	0.0262
0.7083	0.7083	1.2094	1.1003	0.9830	1.0188	0.0358
0.7917	0.7917	1.5954	1.4649	1.3193	1.3833	0.0640
0.8333	0.8333	1.8151	1.6743	1.5143	1.5979	0.0836
0.9167	0.9167	2.3117	2.1523	1.9636	2.1015	0.1379
1.0000	1.0000	2.8902	2.7161	2.5000	2.7183	0.2183

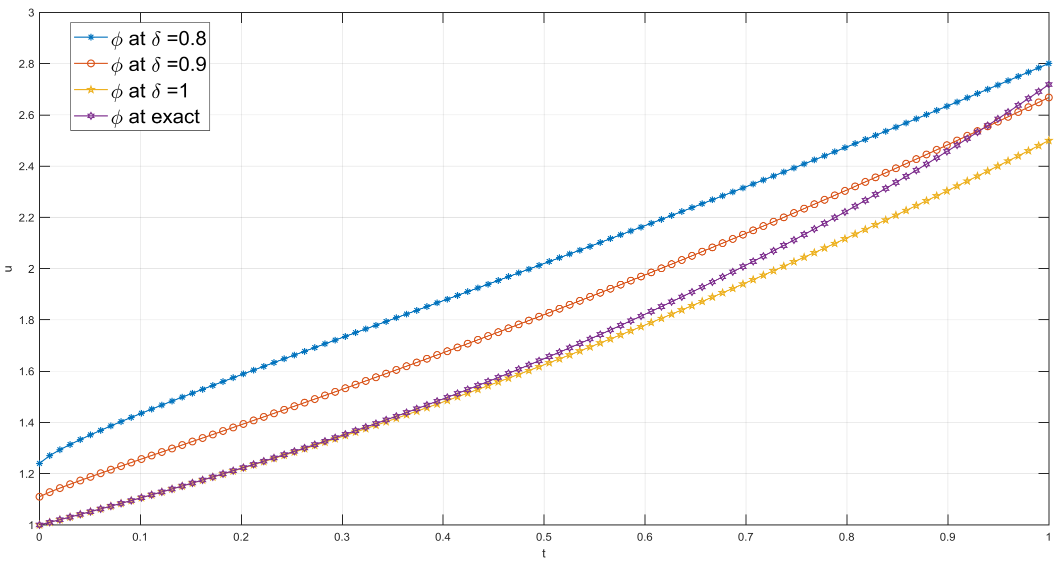

Fig1. The graphs of Eqs.(31,32,37,38,39) at τ,α and $\mu = 1$.

Example 2 Consider the nonlinear equation with the ABCO

$$D_{\tau}^{\alpha}u(\mu,\tau)+\frac{1}{2}(u^2)_{\mu}-u+u^2=0, 0<\delta\leq 1 \ (28)$$

with the initial condition $u(\mu,0) = e^{-\mu}$.

Below we present the LVIM,

using algorithm of the method, we get

$$u_{n+1}=e^{-\mu}-L^{-1}\left((1-\alpha+\alpha s^{\alpha})L\left(u^2+\frac{1}{2}(u^2)_{\mu}-u\right)\right), (29)$$

Now, we discover the approximate solutions as follows:

$$u_0=e^{-\mu},$$

$$u_1=e^{-\mu}-\left((1-\alpha+\alpha s^{\alpha})L(-e^{-\mu})\right)$$

$$= e^{-\mu}\left(2-\alpha+\alpha\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right),$$

$$u_2=e^{-\mu}-\left((1-\alpha+\alpha s^{\alpha})L\left(-e^{-\mu}\left(2-\alpha+\alpha\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right)\right)\right)$$

$$= e^{-\mu}\left((3-3\alpha+\alpha^2)+(3\alpha-2\alpha^2)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\delta^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right), (30)$$

thus, the approximate solution of Eq.(28) can be written,

$$u(\mu,\tau)= e^{-\mu}\left((3-3\alpha+\alpha^2)+(3\alpha-2\alpha^2)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\delta^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right), (31)$$

when choosing $\alpha=1$ in Eq.(31), it becomes

$$u(\mu,\tau)=e^{-\mu}\left(1+\tau+\frac{\tau^2}{2!}+\cdots\right), (32)$$

Now, we present the LHPM,

applying the LHPM to Eq.(19), we get

$$L\left[{}^{\mathcal{ABC}}\mathcal{D}_{\tau}^{\alpha}u(\mu,\tau)=-\frac{1}{2}(u^2)_{\mu}+u-u^2\right], (33)$$

by using the inverse Laplace transform to both sides of Eq.(33) and the initial condition,

$$u(\mu,\tau)=\mu^2+L^{-1}\left[(1-\alpha+\alpha s^{\alpha})L\left[-\frac{1}{2}(u^2)_{\mu}-u^2+u\right]\right], (34)$$

by applying homotopy permutation method on Eq.(34),

$$\sum_{n=0}^{\infty}\wp^n u_n=\mu^2-\wp L^{-1}\left[(1-\alpha+\alpha s^{\alpha})L\left[\sum_{n=0}^{\infty}\wp^n\mathcal{A}_n-\sum_{n=0}^{\infty}\wp^n B_n+\sum_{n=0}^{\infty}\wp^n u_n\right]\right], (35)$$

by comparing both sides of the Eq.(35), the following result is obtained,

$$\wp^0: u_0=e^{-\mu},$$

$$\wp^1: u_1=L^{-1}\left[(1-\alpha+\alpha s^{\alpha})L[\wp^0\mathcal{A}_0-\wp^0 B_0+\wp^0 u_0]\right],$$

$$\wp^2 : u_2 = L^{-1}\left[(1-\alpha+\alpha s^{\alpha})L[\wp^1\mathcal{A}_1 - \wp^1\mathrm{B}_1 + \wp^1 u_1]\right].$$

By the above algorithms,

$$u_0 = \mathrm{e}^{-\mu},$$

$$u_1 = \mathrm{e}^{-\mu}\left(1-\alpha+\alpha\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right),$$

$$u_2 = \mathrm{e}^{-\mu}\left((1-2\delta+\delta^2)+(2\delta-2\delta^2)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\delta^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\right), (36)$$

and so on.

Therefore, the series solution $u(\mu,\tau)$ of Eq. (28) is given by

$$u(\mu,\tau) = \mathrm{e}^{-\mu}\left[(3-3\alpha+\alpha^2)+(3\alpha-2\alpha^2)\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\delta^2\frac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots\right]. (37)$$

If we put $\alpha \to 1$ in Eq.(37) , we obtain

$$u(\mu,\tau) = \mathrm{e}^{-\mu}\left(1+\frac{\tau}{1!}+\frac{\tau^2}{2!}+\cdots\right) = \mathrm{e}^{-\mu}\sum_{k=0}^{\infty}\frac{\tau^k}{k!} = \exp(-\mu+\tau). (38)$$

Ultimately, the exact solution of Eq.(28).

$$u(\mu,\tau) = \mathrm{e}^{-\mu+\tau} \ (39)$$

Tab.2 The values of Eqs.(31,32,37,38,39) at different values of $\tau, \mu,$ *and* α

μ	τ	$u_{\alpha=0.8}$	$u_{\alpha=0.9}$	$u_{\alpha=1}$	u_{exact}	$\lvert u_e - u_1 \rvert$
0.0417	0.0417	1.2828	1.1279	1.0000	1.0000	0.0000
0.1250	0.1250	1.3095	1.1422	0.9997	1.0000	0.0003
0.2083	0.2083	1.3147	1.1468	0.9987	1.0000	0.0013
0.2500	0.2500	1.3126	1.1466	0.9978	1.0000	0.0022
0.3333	0.3333	1.3016	1.1426	0.9952	1.0000	0.0048
0.4167	0.4167	1.2837	1.1343	0.9911	1.0000	0.0089
0.5000	0.5000	1.2606	1.1224	0.9856	1.0000	0.0144
0.5833	0.5833	1.2334	1.1074	0.9785	1.0000	0.0215
0.6250	0.6250	1.2186	1.0989	0.9743	1.0000	0.0257
0.7083	0.7083	1.1871	1.0800	0.9648	1.0000	0.0352
0.7917	0.7917	1.1534	1.0590	0.9538	1.0000	0.0462
0.8750	0.8750	1.1181	1.0362	0.9412	1.0000	0.0588
0.9167	0.9167	1.1000	1.0242	0.9344	1.0000	0.0656
1.0000	1.0000	1.0632	0.9992	0.9197	1.0000	0.0803

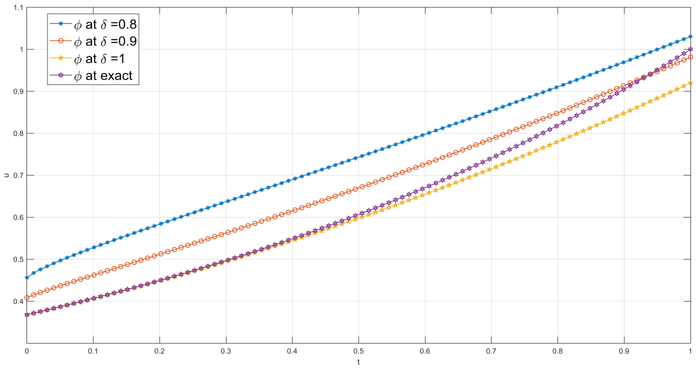

Fig2. The graphs of Eqs.(31,32,37,38,39) among different values of τ,α and $\mu = 1$.

Example 3 Suppose that the nonlinear system with the ABCO:

$$D_\tau^\alpha u(\mu,\tau) - v_\mu + v + u = 0, 0 < \alpha \leq 1,$$

$$D_\tau^\gamma v(\mu,\tau) - u_\mu$$

where $0 < \alpha, \lambda \leq 1$ with the initial condition

$$u(\mu,0) = \sinh(\mu),$$

$$v(\mu,0) = \cosh(\mu)\,.(41)$$

Below we present the LVIM,

We obtain using the method's algorithm

$$u_{n+1}(\mu,\tau) = u(\mu,0) - \mathrm{L}^{-1}\left((1-\alpha+\alpha s^\alpha)\mathrm{L}\{v_{\mu n} - v_n - u_n\}\right),$$

$$v_{n+1}(\mu,\tau) = v(\mu,0) + \mathrm{L}^{-1}\left((1-\lambda+\lambda s^\lambda)\mathrm{L}\{u_{\mu n} - v_n - u_n\}\right), (42)$$

Now we discover the approximate solutions as follows:

$$u_0 = \sinh(\mu)\,, v_0 = \cosh(\mu),$$

$$u_1 = \sinh(\mu) - \cosh(\mu)\left(1-\alpha+\alpha\frac{\tau^\alpha}{\Gamma(\alpha+1)}\right),$$

$$v_1 = \cosh(\mu) - \sinh(\mu)\left(1-\lambda+\lambda\frac{\tau^\lambda}{\Gamma(\lambda+1)}\right),$$

$$u_2 = \sinh(\mu) + \begin{bmatrix} (1-\alpha)(1-\lambda)+\lambda(1-\alpha)\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} \\ +\alpha(1-\lambda)\dfrac{\tau^\alpha}{\Gamma(\alpha+1)} + \alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)} \end{bmatrix}(\sinh(\mu)-\cosh(\mu)) + \begin{bmatrix} -\alpha(1-\alpha)+\alpha(1-\alpha)\dfrac{\tau^\alpha}{\Gamma(\alpha+1)} \\ -\alpha^2\dfrac{\tau^\alpha}{\Gamma(\alpha+1)} + \alpha^2\dfrac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} \end{bmatrix}\cosh(\mu),$$

$$v_2 = \cosh(\mu) + \begin{bmatrix} (1-\alpha)(1-\lambda)+\alpha(1-\lambda)\dfrac{\tau^\alpha}{\Gamma(\alpha+1)} \\ +\lambda(1-\alpha)\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} + \alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)} \end{bmatrix}(\cosh(\mu)-\sinh(\mu)) + \begin{bmatrix} -\lambda(1-\lambda)+\lambda(1-\lambda)\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} \\ -\lambda^2\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} + \lambda^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)} \end{bmatrix}\sinh(\mu).$$

Hence, Eq.(40) has approximate solution is given by

$$u = \sinh(\mu) + \begin{bmatrix} (1-\alpha)(1-\lambda)+\lambda(1-\alpha)\dfrac{\tau^\lambda}{\Gamma(\lambda+1)} \\ +\alpha(1-\lambda)\dfrac{\tau^\alpha}{\Gamma(\alpha+1)} + \alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)} \end{bmatrix}(\sinh(\mu)-\cosh(\mu)) + \begin{bmatrix} -\alpha(1-\alpha)+\alpha(1-\alpha)\dfrac{\tau^\alpha}{\Gamma(\alpha+1)} \\ -\alpha^2\dfrac{\tau^\alpha}{\Gamma(\alpha+1)} + \alpha^2\dfrac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} \end{bmatrix}\cosh(\mu) - \cdots,$$

$$v=\cosh(\mu)+\begin{bmatrix}(1-\alpha)(1-\lambda)+\alpha(1-\lambda)\dfrac{\tau^{\alpha}}{\Gamma(\alpha+1)}\\+\lambda(1-\alpha)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)}+\alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)}\end{bmatrix}(\cosh(\mu)-\sinh(\mu))+\begin{bmatrix}-\lambda(1-\lambda)+\lambda(1-\lambda)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)}\\-\lambda^2\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)}+\lambda^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)}\end{bmatrix}\sinh(\mu)-\cdots.(43)$$

If we put $\alpha\to 1$ and $\lambda\to 1$ in Eq.(43), we recreate the problem solution as below.

$$u(\mu,\tau)=\sinh(\mu)\left(1+\frac{\tau^2}{2!}+\cdots\right)-\cosh(\mu)\left(\tau+\frac{\tau^3}{3!}+\cdots\right),$$

$$v(\mu,\tau)=\cosh(\mu)\left(\tau+\frac{\tau^3}{3!}+\cdots\right)-\sinh(\mu)\left(1+\frac{\tau^2}{2!}+\cdots\right).(44)$$

Now, we present the LHPM,

applying the LHPM to Eq.(40), we get

$$\sum_{n=0}^{\infty}\wp^{n}u_n(\mu,\tau)=\sinh(\mu)-\wp L^{-1}\left[\left(1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}\right)\mathrm{L}\{v_{\mu n}-v_n-u_n\}\right],$$

$$\sum_{n=0}^{\infty}\wp^{n}v_n(\mu,\tau)=\cosh(\mu)-\wp L^{-1}\left[\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathrm{L}\{u_{\mu n}-v_n-u_n\}\right],(45)$$

on comparing both sides of the (45) ,

$$\wp^0: u_0=u(x,0),$$

$$\wp^0: v_0=v(x,0),$$

$$\wp^1: u_1=-L^{-1}\left\{\left(1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}\right)\mathrm{L}\{v_{\mu 0}-v_0-u_0\}\right\},$$

$$\wp^1: v_1=-L^{-1}\left\{\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathrm{L}\{u_{\mu 0}-v_0-u_0\}\right\},$$

$$\wp^2: u_2=-\mathrm{L}^{-1}\left\{\left(1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}\right)\mathrm{L}\{v_{\mu 1}-v_1-u_1\}\right\},$$

$$\wp^2: v_2=-\mathrm{L}^{-1}\left\{\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathrm{L}\{u_{\mu 1}-v_1-u_1\}\right\}.$$

By the above algorithms,

$$u_0=\sinh(\mu),$$

$$v_0=\cosh(\mu),$$

$$u_1=-\cosh(\mu)\left(1-\alpha+\alpha\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right),$$

$$v_1=-\sinh(\mu)\left(1-\lambda+\lambda\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}\right),$$

$$u_2=\begin{bmatrix}(1-\alpha)(1-\lambda)+\lambda(1-\alpha)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)}\\+\alpha(1-\lambda)\dfrac{\tau^{\alpha}}{\Gamma(\alpha+1)}+\alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)}\end{bmatrix}(\sinh(\mu)-\cosh(\mu))+\begin{bmatrix}(1-\alpha)^2+2\alpha(1-\alpha)\dfrac{\tau^{\alpha}}{\Gamma(\alpha+1)}\\+\alpha^2\dfrac{\tau^{2\alpha}}{\Gamma(2\alpha+1)}\end{bmatrix}\cosh(\mu),$$

$$v_2 = \begin{bmatrix} (1-\alpha)(1-\lambda) + \alpha(1-\lambda)\dfrac{\tau^{\alpha}}{\Gamma(\alpha+1)} \\ +\lambda(1-\alpha)\ \dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)} \end{bmatrix} (\cosh(\mu) - \sinh(\mu)) + \begin{bmatrix} (1-\lambda)^2 + 2\lambda(1-\lambda)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ +\lambda^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)} \end{bmatrix} \sinh(\mu).$$

Therefore, the approximate solution of Eq.(40) is given by

$$u = \sinh(\mu) - \cosh(\mu)\left(1 - \alpha + \alpha\frac{\tau^{\alpha}}{\Gamma(\alpha+1)}\right) + (\sinh(\mu) - \cosh(\mu))\begin{bmatrix} (1-\alpha)(1-\lambda) + \lambda(1-\alpha)\ \dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ +\alpha(1-\lambda)\dfrac{\tau^{\alpha}}{\Gamma(\alpha+1)} + \alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)} \end{bmatrix}$$

$$+ \cosh(\mu)\begin{bmatrix} (1-\alpha)^2 + 2\alpha(1-\alpha)\dfrac{\tau^{\alpha}}{\Gamma(\alpha+1)} \\ +\alpha^2\dfrac{\tau^{2\alpha}}{\Gamma(2\alpha+1)} \end{bmatrix} + \cdots,$$

$$v = \cosh(\mu) - \sinh(\mu)\left(1 - \lambda + \lambda\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}\right) + (\cosh(\mu) - \sinh(\mu))\begin{bmatrix} (1-\alpha)(1-\lambda) + \alpha(1-\lambda)\dfrac{\tau^{\alpha}}{\Gamma(\alpha+1)} \\ +\lambda(1-\alpha)\ \dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \alpha\lambda\dfrac{\tau^{\alpha+\lambda}}{\Gamma(\alpha+\lambda+1)} \end{bmatrix}$$

$$+ \sinh(\mu)\begin{bmatrix} (1-\lambda)^2 + 2\lambda(1-\lambda)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ +\alpha^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)} \end{bmatrix} + \cdots. (46)$$

If we put $\alpha \to 1$ and $\lambda \to 1$ in Eq.(46), we get

$$u(\mu,\tau) = \sinh(\mu)\left(1 + \frac{\tau^2}{2!} + \cdots\right) - \cosh(\mu)\left(\tau + \frac{\tau^3}{3!} + \cdots\right),$$

$$v(\mu,\tau) = \cosh(\mu)\left(1 + \frac{\tau^2}{2!} + \cdots\right) - \sinh(\mu)\left(\tau + \frac{\tau^3}{3!} + \cdots\right). (47)$$

This solution is equivalent to the exact solution in closed form:

$$u(\mu,\tau) = \sinh(\mu)\ \cosh(\tau) - \cosh(\mu)\ \sinh(\tau),$$

$$v(\mu,\tau) = \cosh(\mu)\cosh(\mu) - \sinh(\mu)\sinh(\mu). (48)$$

Tab.3 The values of Eq.(43) at different values of $\tau, \mu, and\ \alpha, \lambda$.

μ	τ	$u_{0.8}$	$u_{0.9}$	u_1	u_{exact}	$\lvert u_e - u_1\rvert$
0.0417	0.1375	-0.3289	-0.2132	-0.1042	-0.0960	0.0082
0.1667	0.2500	-0.2888	-0.1891	-0.1099	-0.0834	0.0264
0.2500	0.3250	-0.2500	-0.1657	-0.1192	-0.0751	0.0441
0.3333	0.4000	-0.2028	-0.1365	-0.1332	-0.0667	0.0665
0.4167	0.4750	-0.1473	-0.1014	-0.1521	-0.0584	0.0938
0.5000	0.5500	-0.0833	-0.0598	-0.1766	-0.0500	0.1266
0.5833	0.6250	-0.0105	-0.0111	-0.2073	-0.0417	0.1656
0.6667	0.7000	0.0718	0.0456	-0.2449	-0.0333	0.2116
0.7500	0.7750	0.1646	0.1114	-0.2905	-0.0250	0.2655
0.8333	0.8500	0.2690	0.1874	-0.3453	-0.0167	0.3286
0.9167	0.9250	0.3863	0.2751	-0.4104	-0.0083	0.4021
1.0000	1.0000	0.5181	0.3764	-0.4875	0	0.4875

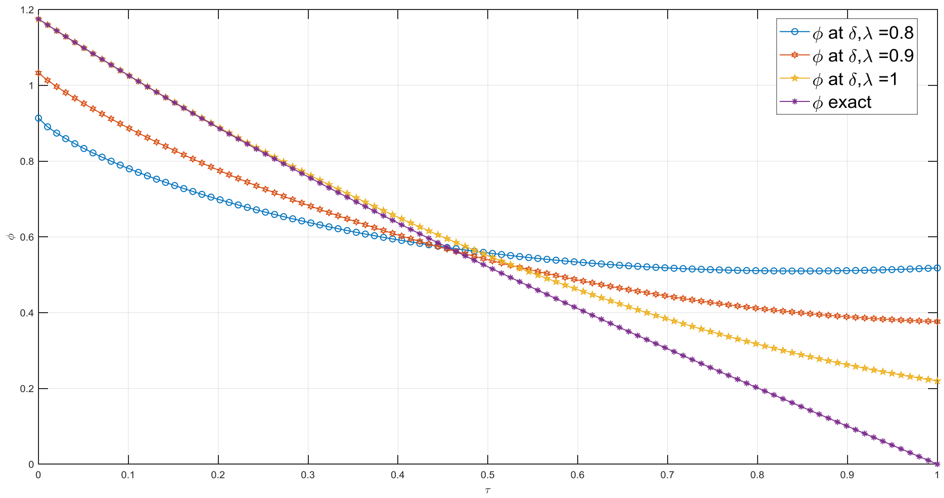

Fig3. The graphs of Eq.(43) among different values of τ,α,λ and $\mu = 1$.

Tab.4 The values of Eq.(46) at different values of $\tau, \mu, and\ \alpha, \lambda$.

μ	τ	$v_{0.8}$	$v_{0.9}$	v_1	v_{exact}	$\lvert v_e - v_1 \rvert$
0.0417	0.1375	1.0046	1.0452	1.0042	1.0046	0.0004
0.1667	0.2500	1.0037	1.0572	0.9989	1.0035	0.0045
0.2500	0.3250	1.0038	1.0666	0.9916	1.0028	0.0113
0.3333	0.4000	1.0047	1.0781	0.9798	1.0022	0.0224
0.4167	0.4750	1.0071	1.0926	0.9628	1.0017	0.0389
0.5000	0.5500	1.0116	1.1110	0.9393	1.0013	0.0619
0.5833	0.6250	1.0189	1.1345	0.9084	1.0009	0.0924
0.6667	0.7000	1.0301	1.1643	0.8690	1.0006	0.1316
0.7500	0.7750	1.0462	1.2016	0.8198	1.0003	0.1805
0.8333	0.8500	1.0687	1.2481	0.7597	1.0001	0.2405
0.9167	0.9250	1.0991	1.3054	0.6872	1.0000	0.3129
1.0000	1.0000	1.1394	1.3756	0.6008	1.0000	0.3992

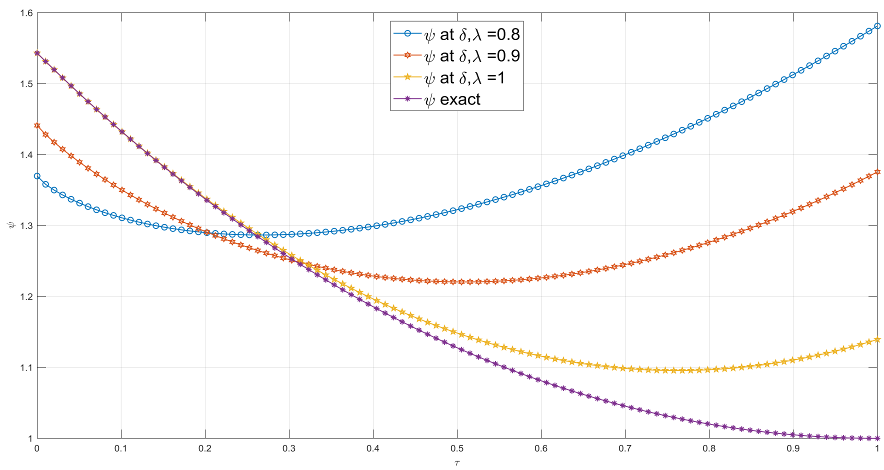

Fig4. The graphs of Eq.(46) among different values of τ,α,λ and $\mu = 1$.

Conclusions

In this study, we have introduced and compared two techniques, the Laplace Variational Iteration Method (LVIM) and the Laplace Homotopy Permutation Method (LHPM), for obtaining approximate solutions to fractional differential equations (FDEs) featuring the Atangana-Baleanu fractional derivative in the Caputo sense. Through a comprehensive analysis of these methods, we have gained valuable insights into their strengths and applicability in addressing this specific class of equations.

Our examination of the LVIM and LHPM has revealed their capacity to provide accurate and practical approximate solutions for FDEs with the Atangana-Baleanu fractional derivative. The LVIM capitalizes on the synergy between the Laplace transform and the variational iteration technique, allowing for systematic approximation of solutions. Similarly, the LHPM's integration of the Laplace transform with the homotopy permutation approach demonstrates its efficacy in handling the same type of equations.

The comparative assessment presented in this paper has highlighted the distinct attributes of both methods. While the LVIM offers an elegant framework for iterative refinement of solutions, the LHPM exhibits its strengths in permutation-based solutions, enabling the exploration of different solution paths. The decision to select between these methods could be influenced by the specific problem context and computational resources available.

As fractional calculus continues to find application in diverse scientific and engineering disciplines, the findings of this study serve as a guide for researchers and practitioners when choosing appropriate techniques for solving FDEs with the Atangana-Baleanu fractional derivative. Moreover, the successful application of LVIM and LHPM in this study indicates their potential contributions to advancing the state-of-the-art in tackling complex fractional differential equations.

In conclusion, the LVIM and LHPM have demonstrated their ability to successfully approximate solutions for FDEs involving the Atangana-Baleanu fractional derivative. This research paves the way for further investigations, refinements, and extensions of these techniques, ultimately leading to enhanced methods for solving intricate mathematical models that arise in various scientific and engineering domains.

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