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APPROXIMATE SOLUTIONS OF FPDES WITH FRACTIONAL DERIVATIVE OF ANTAGANA-BALEANU IN CAPUTO SENSE

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Abstract

The natural variational iteration technique (NVIM) and the natural homotopy permutation method (NHPM) are described in this article to provide an approximation solution to fractional differential equations using the fractional derivative of Atangana Baleaneau in Caputo sense. The algorithms of the two ways were compared, and the results of the two methods were shown to be successful for obtaining the approximate solution for this sort of fractional differential equation.

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Introduction

Fractional calculus and its applications have attracted the attention of academics in recent years, not only in mathematics but also in a wide range of other sciences, including physics, thermodynamics, engineering, economics, etc. Electrical, electrochemistry, statistics, and probability are just a few of the fields where fractional calculus is used extensively. Furthermore, many cosmic events that conventional differential equations are unable to describe can be described by fractional differential equations [1-3].

One of the key techniques for estimating the solution to differential equations is the homotopy permutation approach, which has been the subject of several studies. One of the integrals transforms utilized in the solution of differential equations is the natural transform. To discover the approximate solution to differential equations, we may now combine the homotopy permutation with the natural transform. [4-6].

Researchers in this discipline have thrived from their initial steps, due to numerous applications in physics, economics, engineering, and biological sciences. Several complicated and efficient techniques, such as [7-34], have been created and developed to solve fractional differential equations. The goal of this paper is to provide both approaches for solving FPDEs using the fractional Atangana-Baleanu-Caputo operator (FABCO), NVIM, and NHPM.

In this paper, we apply NVIM, and NHPM to find solution of fractional differential equations with the fractional operator Atangana-Baleanu. The order of the paper is as follows: The basic definitions for calculus and fractional integration are presented in section 2, the methods used are analyzed in section 3, many examples are given that explain the effectiveness of the method proposed in section 4, and finally, the conclusion is provided in section 5.

Basic concepts

Definition 1. [35] The Atangana-Baleanu-Caputo operator when $f \in H^1(\varepsilon 1, \varepsilon 2), \varepsilon 1 > \varepsilon 2$ and for $0 < \mathcal{S} \leq 1$ is

$$^{ABC}D_\tau^{\mathcal{S}} f(\tau) = \frac{B(\mathcal{S})}{1-\mathcal{S}} \int_0^\tau f'(x)\, E_{\mathcal{S}}\left(-\frac{\mathcal{S}}{1-\mathcal{S}}(\tau - x)^{\mathcal{S}}\right) dx, \quad (1)$$

where $B(\mathcal{S})$ is a function such that $\mathfrak{B}(0) = \mathfrak{B}(1) = 1$.

Definition 2. [36] Assume set of function,

$$A = \left\{ f(\tau) \middle| \exists M, \alpha_1, \alpha_2 > 0, |f(\tau)| < M e^{\frac{|\tau|}{\alpha_j}}, \tau \in (-1)^j \times [0, \infty) \right\},$$

Then the natural transform over A set

$$N(f(\tau)) = R(u,s) = \int_0^\infty e^{-st} f(ut)\, dt. \quad (2)$$

The natural transform yields the Laplace transform via the following formula[37],

$$R(u,s) = \frac{1}{u} \int_0^\infty e^{-st/u} f(t)\, dt = \frac{1}{u} F\left(\frac{s}{u}\right). \quad (3)$$

From [38] and Eq.(3) , we get this relation

$$N\left(^{ABC}D_\tau^{\mathcal{S}} f(\tau)\right) = \frac{B(\mathcal{S})}{1 - \mathcal{S} + \mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}} \left(R(u,s) - \frac{1}{s} f(0)\right) \quad (4)$$

Definition 3. [37] The inverse natural transform of a function is defined by

$$N^{-1}(R(u,s)) = f(t) = \frac{1}{2i\pi} \int_{p-i\infty}^{p+i\infty} e^{\frac{st}{u}} R(u,s) dt, \quad (5)$$

where s and u are natural transform variables and p is a real constant.

Analysis of NVIM

Let's considering FPDEs with FABCO

$$^{ABC}D_\tau^{\mathcal{S}} \Phi(\mu,\tau) + L(\Phi(\mu,\tau)) + M(\Phi(\mu,\tau)) = f(\mu,\tau), \quad (6)$$

with initial condition $\Phi(\mu, 0) = \Phi_0(\mu)$, where $^{ABC}D_\tau^{\mathcal{S}}$ is the FABCO, L is the linear operator, M is the nonlinear operator and $f(\mu, \tau)$ is a source term.

We obtain by applying the natural transform to Eq.(6) with the stated initial condition

$$\frac{B(\mathcal{S})}{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}} \left(N(\Phi) - \frac{1}{s}\Phi(\mu,0)\right) = N[f(\mu,\tau) - L(\Phi) - M(\Phi)], \quad (7)$$

by substituting initial condition of Eq.(8),

$$\bar{\Phi} = \frac{1}{s}\Phi_0(\mu) - \frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathfrak{B}(\mathcal{S})} N[f(\mu,\tau) - L(\Phi) - M(\Phi)], \qquad (8)$$

appling variation intretion method

$$\bar{\Phi}_{n+1} = \bar{\Phi}_n + \lambda\left(\bar{\Phi}_n - \frac{1}{s}\Phi_0(\mu) + \frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathfrak{B}(\mathcal{S})} N[L(\Phi_n) + M(\Phi_n) - f(\mu,\tau)]\right), \qquad (9)$$

Note that λ is the Lagrange multiplier, and since $0 < \mathcal{S} < 1$, hence $\lambda = -1$, we obtain after applying the inverse of the natural transform to both sides of the equation

$$\Phi_{n+1} = \Phi_0(\mu) - N^{-1}\left(\frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathfrak{B}(\mathcal{S})} N[L(\Phi_n) + M(\Phi_n) - f(\mu,\tau)]\right), \qquad (10)$$

where we have the initial iteration is $\Phi(\mu,0) = \Phi_0(\mu)$, and hence

$\Phi(\mu,\tau) = \lim_{k\to\infty} \Phi_k(\mu,\tau)$.

Analysis of NHPM

Subject to the provided initial condition, apply the natural transform to Eq.(6).

$$\frac{\mathcal{B}(\mathcal{S})}{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}\left(\mathcal{N}(u,s) - \frac{1}{s}v(0)\right) = \mathcal{N}(g(x,t) - \mathcal{L}[v(x,t)] - \mathcal{M}\,[v(x,t)]), \qquad (11)$$

by substituting initial condition of Eq.(11),

$$\bar{v} = \frac{1}{s}v_0(x) - \frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathcal{B}(\mathcal{S})}\mathcal{N}(\mathcal{L}[v] + \mathcal{M}\,[v] - g), \qquad (12)$$

Taking the inverse of the natural transform and applying it to both sides of the equation (12),

$$v = v_0(x) + \mathcal{N}^{-1}\left(\frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathcal{B}(\mathcal{S})}\mathcal{N}(g)\right) - \mathcal{N}^{-1}\left(\frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathcal{B}(\mathcal{S})}\mathcal{N}(\mathcal{L}[v] + \mathcal{M}\,[v])\right), \qquad (13)$$

applying homotopy permutation method,

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t)\,, \mathcal{N}[u(x,t)] = \sum_{n=0}^{\infty} p^n \mathcal{H}_n(v), \qquad (14)$$

where

$$\mathcal{H}_n(v_1, v_2, v_3, \dots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}[\mathcal{N}(\sum_{i=0}^{n} p^i v_i(x,t))]_{p=0}\,, n = 0,1,2,\dots.$$

Substituting Eq.(14) into Eq.(13) gives us the result that,

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = \mathcal{G}(x,t) - p\left(\mathcal{N}^{-1}\left(\frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathcal{B}(\mathcal{S})}\mathcal{N}(\sum_{n=0}^{\infty} p^n \mathcal{L}[v_n] + \sum_{n=0}^{\infty} p^n \mathcal{H}_n(v))\right)\right), \qquad (15)$$

on comparing both sides of Eq.(15), the following result is obtained,

$$\mathcal{p}^0: \mathcal{v}_0(x,t) = \mathcal{G}(x,t),$$

$$\mathcal{p}^1: \mathcal{v}_1(x,t) = -\mathcal{N}^{-1}\left(\frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathcal{B}(\mathcal{S})}\mathcal{N}\left(\mathcal{L}[\mathcal{v}_0]+\mathcal{H}_0(\mathcal{v})\right)\right), \vdots$$

$$\mathcal{p}^n: \mathcal{v}_n(x,t) = -\mathcal{N}^{-1}\left(\frac{1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}}{\mathcal{B}(\mathcal{S})}\mathcal{N}\left(\mathcal{L}[\mathcal{v}_{n-1}]+\mathcal{H}_{n-1}(\mathcal{v})\right)\right), \qquad (16)$$

using the parameter $\mathcal{p}$, we expand the solution in the following form

$$\mathcal{v}(x,t) = \sum_{n=0}^{\infty} \mathcal{p}^n \mathcal{v}_n(x,t), \qquad (17)$$

choose p = 1 results in the solution of Eq.(11)

$$\mathcal{v}(x,t) = \lim_{\mathcal{p}\to 1} \sum_{n=0}^{\infty} \mathcal{p}^n \mathcal{v}_n(x,t) = \sum_{n=0}^{\infty} \mathcal{v}_n(x,t). \qquad (18)$$

Application

In this part, two nonlinear equations and a linear system will be solved using the NVIM and NHPM techniques, with the assumption that $\mathfrak{B}(\mathcal{S}) = 1$.

Example 1. Consider the nonlinear equation with the FABCO

$${}^{\mathcal{ABC}}\mathcal{D}_{\tau}^{\mathcal{S}}\Phi(\mu,\tau) = -\frac{\partial}{\partial\mu}\left(\frac{12}{\mu}\Phi-\mu\right)\Phi + \frac{\partial^2}{\partial\mu^2}\Phi^2, 0 < \delta \le 1, \quad (19)$$

depending on the initial condition $\Phi(\mu,0) = \mu^2$.

Below we present the NVIM,

Applying the NVIM to Eq.(19), we get

$$\Phi_{n+1} = \mu^2 - N^{-1}\left(\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)N\left(\frac{12}{\mu}\Phi_{\mu n}\Phi_n - \frac{12}{\mu^2}\Phi_n^2 - (\Phi_n^2)_{\mu\mu} + \Phi_n\right)\right), \qquad (20)$$

now, we find the approximate solutions as,

$$\Phi_0 = \mu^2,$$

$$\Phi_1 = \mu^2 + N^{-1}\left(\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)N(\mu^2)\right) = \mu^2\left((2-\mathcal{S})+\mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right),$$

$$\Phi_2 = \mu^2 + N^{-1}\left(\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)N\left((2-\mathcal{S})+\mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right)\right)$$

$$= \mu^2\left(1+(1-\mathcal{S})(2-\mathcal{S})+(\mathcal{S}(1-\mathcal{S})+\mathcal{S}(2-\mathcal{S}))\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\mathcal{S}^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}\right), \qquad (21)$$

and so on.

Therefore, the series solution $\Phi(\mu,\tau)$ of Eq.(19) is given by

$$\Phi(\mu,\tau)=\mu^2\left((\mathcal{S}^2-3\mathcal{S}+3)+(3\mathcal{S}-2\mathcal{S}^2)\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\mathcal{S}^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}+\cdots\right), \quad (22)$$

when choosing $\mathcal{S}=1$ in Eq.(22), it becomes

$$\Phi(\mu,\tau)=\mu^2\left(1+\tau+\frac{\tau^2}{2!}+\cdots\right), \quad (23)$$

Now, we present the NHPM,

applying the NHPM to Eq.(19), we get

$$\sum_{n=0}^{\infty}\wp^n\Phi_n=\mu^2-\wp\mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}\left[\begin{matrix}\sum_{n=0}^{\infty}\wp^n\mathcal{A}_n-\sum_{n=0}^{\infty}\wp^n\mathcal{B}_n\\+\sum_{n=0}^{\infty}\wp^n\mathcal{C}_n+\sum_{n=0}^{\infty}\wp^n\Phi_n\end{matrix}\right]\right], \quad (24)$$

by comparing both sides of the Eq.(24), the following result is obtained,

$$\wp^0:\Phi_0=\mu^2,$$

$$\wp^1:\Phi_1=\mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}[\wp^0\mathcal{A}_0-\wp^0\mathcal{B}_0+\wp^0\mathcal{C}_0+\wp^0\Phi_0]\right],$$

$$\wp^2:\Phi_2=\mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}[\wp^1\mathcal{A}_1-\wp^1\mathcal{B}_1+\wp^1\mathcal{C}_1+\wp^1\Phi_1]\right],$$

by the above algorithms,

$$\Phi_0=\mu^2,$$

$$\Phi_1=\mu^2\left(1-\mathcal{S}+\mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right),$$

$$\Phi_2=\mu^2\left((1-2\delta+\delta^2)+(2\delta-2\delta^2)\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\delta^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}\right),$$

and so on.

Therefore, the series solution$\Phi(\mu,\tau)$ of Eq. (19) is given by

$$\Phi(\mu,\tau)=\mu^2\left[(3-3\mathcal{S}+\mathcal{S}^2)+(3\mathcal{S}-2\mathcal{S}^2)\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\delta^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}+\cdots\right]. \quad (25)$$

If we put $\mathcal{S}\to 1$ in Eq.(25) , we get the approximate and exact solution

$$\Phi(\mu,\tau)=\mu^2\left(1+\frac{\tau}{1!}+\frac{\tau^2}{2!}+\cdots\right). \quad (26)$$

Ultimately, the exact solution of Eq.(19),

$$\Phi(\mu,\tau)=\mu^2e^{\tau}. \quad (27)$$

Example 2. Consider the nonlinear equation with the FABCO

$$D_{\tau}^{\mathcal{S}}\Phi(\mu,\tau)+\frac{1}{2}(\Phi^2)_{\mu}-\Phi+\Phi^2=0, 0<\delta\leq 1 \qquad (28)$$

with the initial condition $\Phi(\mu,0)=\mathrm{e}^{-\mu}$.

Below we present the NVIM,

using algorithm of the method, we get

$$\Phi_{n+1}=\mathrm{e}^{-\mu}-N^{-1}\left(\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)N\left(\Phi^2+\frac{1}{2}(\Phi^2)_{\mu}-\Phi\right)\right), \quad (29)$$

Now, we discover the approximate solutions as follows: $\Phi_0=\mathrm{e}^{-\mu}$,

$$\Phi_1=\mathrm{e}^{-\mu}-\left(\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)N(-\mathrm{e}^{-\mu})\right)=\mathrm{e}^{-\mu}\left(2-\mathcal{S}+\mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right),$$

$$\Phi_2=\mathrm{e}^{-\mu}-\left(\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)N\left(-\mathrm{e}^{-\mu}\left(2-\mathcal{S}+\mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right)\right)\right)$$

$$=\mathrm{e}^{-\mu}\left((3-3\mathcal{S}+\mathcal{S}^2)+(3\mathcal{S}-2\mathcal{S}^2)\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\delta^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}\right), \quad (30)$$

thus, the approximate solution of Eq.(28) can be written,

$$\Phi(\mu,\tau)=\mathrm{e}^{-\mu}\left((3-3\mathcal{S}+\mathcal{S}^2)+(3\mathcal{S}-2\mathcal{S}^2)\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\delta^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}+\cdots\right), \quad (31)$$

when choosing $\mathcal{S}=1$ in Eq.(31), it becomes

$$\Phi(\mu,\tau)=\mathrm{e}^{-\mu}\left(1+\tau+\frac{\tau^2}{2!}+\cdots\right), \quad (32)$$

Now, we present the NHPM,

applying the NHPM to Eq.(19), we get

$$\mathcal{N}\left[{}^{\mathcal{ABC}}\mathcal{D}_{\tau}^{\mathcal{S}}\Phi(\mu,\tau)=-\frac{1}{2}(\Phi^2)_{\mu}+\Phi-\Phi^2\right], \quad (33)$$

by using the inverse natural transform to both sides of Eq.(33) and the initial condition,

$$\Phi(\mu,\tau)=\mu^2+\mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}\left[-\frac{1}{2}(\Phi^2)_{\mu}-\Phi^2+\Phi\right]\right], \quad (34)$$

by applying homotopy permutation method on Eq.(34),

$$\sum_{n=0}^{\infty}p^n\Phi_n=\mu^2-p\mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}\left[\sum_{n=0}^{\infty}p^n\mathcal{A}_n-\sum_{n=0}^{\infty}p^n\mathcal{B}_n+\sum_{n=0}^{\infty}p^n\Phi_n\right]\right], \quad (35)$$

by comparing both sides of the Eq.(35), the following result is obtained,

$\mathscr{p}^0 : \Phi_0 = e^{-\mu}$,

$$\mathscr{p}^1 : \Phi_1 = \mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}[\mathscr{p}^0\mathcal{A}_0 - \mathscr{p}^0\mathcal{B}_0 + \mathscr{p}^0\Phi_0]\right],$$

$$\mathscr{p}^2 : \Phi_2 = \mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}[\mathscr{p}^1\mathcal{A}_1 - \mathscr{p}^1\mathcal{B}_1 + \mathscr{p}^1\Phi_1]\right].$$

By the above algorithms,

$\Phi_0 = e^{-\mu}$,

$\Phi_1 = e^{-\mu}\left(1-\mathcal{S}+\mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right)$,

$$\Phi_2 = e^{-\mu}\left((1-2\delta+\delta^2)+(2\delta-2\delta^2)\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\delta^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}\right), \qquad (36)$$

and so on.

Therefore, the series solution $\Phi(\mu,\tau)$ of Eq. (28) is given by

$$\Phi(\mu,\tau) = e^{-\mu}\left[(3-3\mathcal{S}+\mathcal{S}^2)+(3\mathcal{S}-2\mathcal{S}^2)\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\delta^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}+\cdots\right]. \qquad (37)$$

If we put $\mathcal{S} \to 1$ in Eq.(37) , we obtain

$$\Phi(\mu,\tau) = e^{-\mu}\left(1+\frac{\tau}{1!}+\frac{\tau^2}{2!}+\cdots\right) = e^{-\mu}\sum_{k=0}^{\infty}\frac{\tau^k}{k!} = \exp(-\mu+\tau). \qquad (38)$$

Ultimately, the exact solution of Eq.(28).

$$\Phi(\mu,\tau) = e^{-\mu+\tau} \qquad (39)$$

Example 3. Suppose that the nonlinear system with the FABCO:

$$D_\tau^{\mathcal{S}}\Phi(\mu,\tau) - \Psi_\mu + \Psi + \Phi = 0, 0 < \mathcal{S} \le 1,$$

$$D_\tau^{\gamma}\Psi(\mu,\tau) - \Phi_\mu + \Psi + \Phi = 0, 0 < \gamma \le 1, \qquad (40)$$

where $0 < \mathcal{S}, \lambda \le 1$ with the initial condition

$\Phi(\mu,0) = \sinh(\mu)$,

$\Psi(\mu,0) = \cosh(\mu)$. $\qquad$ (41)

Below we present the NVIM,

We obtain using the method's algorithm

$$\Phi_{n+1}(\mu,\tau) = \Phi(\mu,0) - N^{-1}\left(\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}\{\Psi_{\mu n} - \Psi_n - \Phi_n\}\right),$$

$$\Psi_{n+1}(\mu,\tau) = \Psi(\mu,0) + N^{-1}\left(\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathcal{N}\{\Phi_{\mu n} - \Psi_n - \Phi_n\}\right), \qquad (42)$$

Now we discover the approximate solutions as follows:

$\Phi_0 = \sinh(\mu)\,, \Psi_0 = \cosh(\mu)$,

$$\Phi_1 = \sinh(\mu) - \cosh(\mu)\left(1 - \mathcal{S} + \mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right),$$

$$\Psi_1 = \cosh(\mu) - \sinh(\mu)\left(1 - \lambda + \lambda\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}\right),$$

$$\Phi_2 = \sinh(\mu) + \begin{bmatrix} (1-\mathcal{S})(1-\lambda) + \lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ +\mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} + \mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)} \end{bmatrix}(\sinh(\mu) - \cosh(\mu))$$
$$+ \begin{bmatrix} -\mathcal{S}(1-\mathcal{S}) + \mathcal{S}(1-\mathcal{S})\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} \\ -\mathcal{S}^2\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} + \mathcal{S}^2\dfrac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)} \end{bmatrix}\cosh(\mu),$$

$$\Psi_2 = \cosh(\mu) + \begin{bmatrix} (1-\mathcal{S})(1-\lambda) + \mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} \\ +\lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)} \end{bmatrix}(\cosh(\mu) - \sinh(\mu))$$
$$+ \begin{bmatrix} -\lambda(1-\lambda) + \lambda(1-\lambda)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ -\lambda^2\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \lambda^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)} \end{bmatrix}\sinh(\mu).$$

Hence, Eq.(40) has approximate solution is given by

$$\Phi = \sinh(\mu) + \begin{bmatrix} (1-\mathcal{S})(1-\lambda) + \lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ +\mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} + \mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)} \end{bmatrix}(\sinh(\mu) - \cosh(\mu))$$
$$+ \begin{bmatrix} -\mathcal{S}(1-\mathcal{S}) + \mathcal{S}(1-\mathcal{S})\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} \\ -\mathcal{S}^2\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} + \mathcal{S}^2\dfrac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)} \end{bmatrix}\cosh(\mu) - \cdots,$$

$$\Psi = \cosh(\mu) + \begin{bmatrix} (1-\mathcal{S})(1-\lambda) + \mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} \\ +\lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)} \end{bmatrix}(\cosh(\mu) - \sinh(\mu))$$
$$+ \begin{bmatrix} -\lambda(1-\lambda) + \lambda(1-\lambda)\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ -\lambda^2\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \lambda^2\dfrac{\tau^{2\lambda}}{\Gamma(2\lambda+1)} \end{bmatrix}\sinh(\mu) - \cdots. \quad (43)$$

If we put $\mathcal{S} \to 1$ and $\lambda \to 1$ in Eq.(43), we recreate the problem solution as below.

$$\Phi(\mu,\tau) = \sinh(\mu)\left(1 + \frac{\tau^2}{2!} + \cdots\right) - \cosh(\mu)\left(\tau + \frac{\tau^3}{3!} + \cdots\right),$$

$$\Psi(\mu,\tau) = \cosh(\mu)\left(\tau+\frac{\tau^3}{3!}+\cdots\right)-\sinh(\mu)\left(1+\frac{\tau^2}{2!}+\cdots\right). \quad (44)$$

Now, we present the NHPM,

applying the NHPM to Eq.(40), we get

$$\sum_{n=0}^{\infty}\wp^n\Phi_n(\mu,\tau) = \sinh(\mu)-\wp\mathcal{N}^{-1}\left[\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}\{\Psi_{\mu n}-\Psi_n-\Phi_n\}\right],$$

$$\sum_{n=0}^{\infty}\wp^n\Psi_n(\mu,\tau) = \cosh(\mu)-\wp\mathcal{N}^{-1}\left[\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathcal{N}\{\Phi_{\mu n}-\Psi_n-\Phi_n\}\right], \quad (45)$$

on comparing both sides of the (45) ,

$\wp^0:\Phi_0 = v(x,0)$, $\wp^0:\Psi_0 = \Psi(x,0)$,

$$\wp^1:\Phi_1 = -\mathcal{N}^{-1}\left\{\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}\{\Psi_{\mu 0}-\Psi_0-\Phi_0\}\right\},$$

$$\wp^1:\Psi_1 = -\mathcal{N}^{-1}\left\{\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathcal{N}\{\Phi_{\mu 0}-\Psi_0-\Phi_0\}\right\},$$

$$\wp^2:\Phi_2 = -\mathcal{N}^{-1}\left\{\left(1-\mathcal{S}+\mathcal{S}\left(\frac{u}{s}\right)^{\mathcal{S}}\right)\mathcal{N}\{\Psi_{\mu 1}-\Psi_1-\Phi_1\}\right\},$$

$$\wp^2:\Psi_2 = -\mathcal{N}^{-1}\left\{\left(1-\lambda+\lambda\left(\frac{u}{s}\right)^{\lambda}\right)\mathcal{N}\{\Phi_{\mu 1}-\Psi_1-\Phi_1\}\right\}.$$

By the above algorithms,

$\Phi_0 = \sinh(\mu), \Psi_0 = \cosh(\mu)$,

$$\Phi_1 = -\cosh(\mu)\left(1-\mathcal{S}+\mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right),$$

$$\Psi_1 = -\sinh(\mu)\left(1-\lambda+\lambda\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}\right),$$

$$\Phi_2 = \begin{bmatrix} (1-\mathcal{S})(1-\lambda)+\lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ +\mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)}\end{bmatrix}(\sinh(\mu)-\cosh(\mu)) \\ +\left[(1-\mathcal{S})^2+2\mathcal{S}(1-\mathcal{S})\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}+\mathcal{S}^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}\right]\cosh(\mu)\,,$$

$$\Psi_2 = \begin{bmatrix} (1-\mathcal{S})(1-\lambda)+\mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} \\ +\lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)}+\mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)}\end{bmatrix}(\cosh(\mu)-\sinh(\mu)) \\ +\left[(1-\lambda)^2+2\lambda(1-\lambda)\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}+\lambda^2\frac{\tau^{2\lambda}}{\Gamma(2\lambda+1)}\right]\sinh(\mu)\,.$$

Therefore, the approximate solution of Eq.(40) is given by

$$\Phi = \sinh(\mu) - \cosh(\mu)\left(1 - \mathcal{S} + \mathcal{S}\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)}\right)$$
$$+\begin{bmatrix} (1-\mathcal{S})(1-\lambda) + \lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} \\ +\mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} + \mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)} \end{bmatrix}(\sinh(\mu) - \cosh(\mu))$$
$$+\left[(1-\mathcal{S})^2 + 2\mathcal{S}(1-\mathcal{S})\frac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} + \mathcal{S}^2\frac{\tau^{2\mathcal{S}}}{\Gamma(2\mathcal{S}+1)}\right]\cosh(\mu) + \cdots,$$

$$\Psi = \cosh(\mu) - \sinh(\mu)\left(1 - \lambda + \lambda\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}\right)$$
$$+\begin{bmatrix} (1-\mathcal{S})(1-\lambda) + \mathcal{S}(1-\lambda)\dfrac{\tau^{\mathcal{S}}}{\Gamma(\mathcal{S}+1)} \\ +\lambda(1-\mathcal{S})\dfrac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \mathcal{S}\lambda\dfrac{\tau^{\mathcal{S}+\lambda}}{\Gamma(\mathcal{S}+\lambda+1)} \end{bmatrix}(\cosh(\mu) - \sinh(\mu))$$
$$+\left[(1-\lambda)^2 + 2\lambda(1-\lambda)\frac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \mathcal{S}^2\frac{\tau^{2\lambda}}{\Gamma(2\lambda+1)}\right]\sinh(\mu) + \cdots. (46)$$

If we put $\mathcal{S} \to 1$ and $\lambda \to 1$ in Eq.(46), we get

$$\Phi(\mu,\tau) = \sinh(\mu)\left(1 + \frac{\tau^2}{2!} + \cdots\right) - \cosh(\mu)\left(\tau + \frac{\tau^3}{3!} + \cdots\right),$$
$$\Psi(\mu,\tau) = \cosh(\mu)\left(1 + \frac{\tau^2}{2!} + \cdots\right) - \sinh(\mu)\left(\tau + \frac{\tau^3}{3!} + \cdots\right). (47)$$

This solution is equivalent to the exact solution in closed form:

$$\Phi(\mu,\tau) = \sinh(\mu)\cosh(\tau) - \cosh(\mu)\sinh(\tau),$$
$$\Psi(\mu,\tau) = \cosh(\mu)\cosh(\mu) - \sinh(\mu)\sinh(\mu). (48)$$

Conclusions

In this work, two nonlinear equations and a linear system of partial differential equations were successfully approximatively solved utilizing the NVIM and NHPM employing the FABO sense. The examples show that the NVIM and NHPM conclusions are very similar. The approaches are particularly successful in analytically and numerically solving a wide range of classes of linear and non-linear fractional differential equations.

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