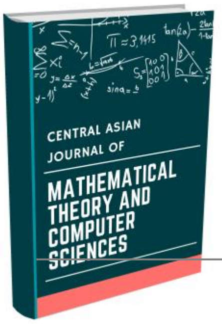

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A MULTIDIMENSIONAL BOUNDARY ANALOGUE OF HARTOGS'S THEOREM FOR INTEGRABLE FUNCTIONS ON n-CIRCULAR DOMAINS

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Abstract

In the present paper we consider integrable functions given on the boundary of n-circular domain D⊂C^n, n>1 and having one-dimensional property of holomorphic extension along the families of complex lines, passing through finite number of points of D. We prove the existence of holomorphic extension of such functions in D.

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Introduction

One of the important problems of complex analysis is holomorphic extension of integrable functions defined on the boundary of a domain $D \subset \mathbb{C}^n$, $n > 1$. Of particular interest is the question of integrable functions with one-dimensional holomorphic extension property along complex lines. On the complex plane $\mathbb{C}$ the problem of one-dimensional holomorphic extension is trivial. So results in this area are essential multidimensional.

The first results concerning this area are due to M. L. Agranovskii and P.E.Val'skii [3], who have studied functions with the one-dimensional holomorphic extension property on a ball. This investigation was based on properties of the group of automorphisms of sphere.

E. L. Stout in [29], using a complex Radon transformation, adopted Agranovskii–Val'skii Theorem to an arbitrary bounded domain with smooth boundary. An alternative proof of Stout's Theorem was given by A.M. Kytmanov in [2], who applied the Bochner–Martinelli Theorem. The idea of using the integral representations (Bochner–Martinelli, Cauchy–Fantappie, logarithmic residue) has been useful in the study of functions with one-dimensional holomorphic continuation property (see review [16]).

The question of finding several families of complex lines which suffice for for holomorphic extension was raised in [12]. Clearly, the family of complex lines passing through one point is not enough. As shown in [17], the family of complex lines passing through a finite number of points also, generally speaking, is not sufficient.

In [17] it was proved that the family of complex lines crossing the germ of a generic manifold γ, is sufficient for the holomorphic extension. In [18] the authors considered continuous functions given on the boundary of a bounded domain D in $\mathbb{C}^n$, $n > 1$, with the one-dimensional holomorphic extension property along families of complex lines. Also studied was the existence of holomorphic extensions of these functions to D depending on the dimension and location of the families of complex lines. Various another families and related problems were studied by many authours [4-7,13]. We note that in papers [5,13] it was shown that a family of complex lines passing through a finite set of points in general position sufficient for holomorphic extension. But it was proved only for real-analytic or infinitely differentiable functions defined on the boundary. In [8,14] were shown that for holomorphic extension of continuous functions it suffices to take a family of complex lines passing through $n+1$ points lying at the interior of ball. Another proof of this result based on applications of integral representation was given in [19, 20]. In [24-26] were considered sufficient conditions for holomorphic extension of integrable functions for a family of complex lines passing through open subset lying in a domain D.

The example of Globevnik [13] shows that for continuous functions on the boundary of ball in $\mathbb{C}^2$ two points is not enough for holomorphic extension. In the paper [21] was considered continuous functions given on the boundary of a ball B of $\mathbb{C}^n, n>1$, having one-dimensional property of holomorphic extension along the families of complex lines, passing through finite number of points of D.

In this paper we generalize this result for integrable functions. In Section 2 we consider the Szegő kernel in n-circular domains. In Sections 3 and 4 we consider the Poisson kernel and modified Poisson kernel on n-circular domains. In Section 5 we will consider integrable functions given on the boundary of n-circular domain $D \subset \mathbb{C}^n, n>1$ and having one-dimensional property of holomorphic extension along the families of complex lines, passing through finite number of points of D. We prove the existence of holomorphic extension of such functions in the domain D (see Theorem 5.10).

1.The Szegő kernel on n-circular domains

Let D be a bounded complete n-circular domain in $\mathbb{C}^n$ with the center at the origin, that is, together with a point $z^0 = (z_1^0, \dots, z_n^0) \in D$ it contains a polydisc

$$\{z \in \mathbb{C}^n\colon |z_k| \le |z_k^0|,\ k=1,\dots,n\}.$$

Denote by $D^+ = \{(|z_1|, \dots, |z_n|)\colon z \in D\}$ the image of a domain D in the absolute octant

$$\mathbb{R}_n^+ = \{(x_1, \dots, x_n)\colon |x_k| \ge 0,\ k=1,\dots,n\}.$$

Let $\partial D^+ = \{(|z_1|, \dots, |z_n|)\colon z \in \partial D\}$.

Consider a finite measure μ on ∂D^+. A measure μ is said to be massive on the Shilov boundary [2, Page 76] if for any subset $E \subset \partial D^+$ with a zero measure μ satisfied a condition $\overline{\partial D^+ \setminus E} \supset S(D^+)$, where $S(D^+)$ is the image of the Shilov boundary $S(D)$ in the absolute octant.

Further we need the following result from [11, Section 3.1].

Proposition 2.1. *If D is a strongly pseudoconvex n-circular domain (i.e., strictly logarithmically convex) then the Shilov boundary $S(D)$ coincides with boundary of the domain.*

Proposition 2.1 implies that the Lebesgue measure μ on the boundary of such domain is massive. From now on we shall assume that μ is a massive measure.

Define the Szegő kernel of domain D:

$$h(\bar\zeta, z) = \sum_{\alpha \ge 0} a_\alpha\, \bar\zeta^\alpha z^\alpha, \qquad (2.1)$$

where

$$a_\alpha = \frac{1}{\int_{\partial D^+} |\zeta|^{2\alpha}\, d\mu} = \frac{1}{\int_{\partial D^+} |\zeta_1|^{2\alpha_1} \cdot \dots \cdot |\zeta_n|^{2\alpha_n} d\mu}$$

and $\alpha = \{\alpha_1, \dots, \alpha_n\}$ is a multi-index such that $\alpha \ge 0$ (i.e., $\alpha_k \ge 0$, $k=1,\dots,n$) and $z^\alpha = z_1^{\alpha_1} \cdot \dots \cdot z_n^{\alpha_n}$, $\| \alpha \| = \alpha_1 + \dots + \alpha_n$.

Recall a definition of a class $\mathcal{H}^p(D)$. A holomorphic function $f \in \mathcal{H}^p(D)$ $(p>0)$, if

$$\sup_{\epsilon > 0} \int_{\partial D} \left| f\big(\zeta - \epsilon \nu(\zeta)\big)\right|^p d\sigma < +\infty,$$

where $d\sigma$ is an element of the surface ∂D and $\nu(\zeta)$ is the outer unit normal vector to the surface ∂D at the point ζ. It is well-known that normal boundary values of $f \in \mathcal{H}^p(D)$ belong to the class $\mathcal{L}^p(\partial D)$ (with respect to the measure $d\sigma$).

The following result gives us the existence of the Szegő kernels on the n-circular domains.

Theorem 2.2. *Let μ be a finite measure on ∂D^+. For any function $f \in \mathcal{H}^p(D)$, $(p \ge 1)$ there exists a Szegő representation*

$$f(z) = \lim_{r \to 1} \frac{1}{(2\pi i)^n} \int_{\partial D^+} d\mu \int_{\Delta_{|\zeta|}} f(\zeta) h(\bar\zeta, rz) \frac{d\zeta}{\zeta}, \qquad z \in D, \qquad (2.2)$$

where $\Delta_{|\zeta|} = \{\zeta\colon \zeta_1 = |\zeta_1| e^{i\theta_1}, \dots, \zeta_n = |\zeta_n| e^{i\theta_n},\ 0 \le \theta_k \le 2\pi,\ k=1,\dots,n,\ |\zeta| \in \partial D^+\}$, $\frac{d\zeta}{\zeta} = \frac{d\zeta_1}{\zeta_1} \wedge \cdot \frac{d\zeta_n}{\zeta_n}$, and the Szegő kernel $h(\bar\zeta, z) = h(\bar\zeta_1 z_1, \dots, \bar\zeta_n z_n)$ as a function of $\bar\zeta$ lie in $\mathcal{O}(\overline{D})$ for fixed $z \in D$, and as a function of z lie in $\mathcal{O}(D)$ for fixed $\zeta \in \partial D$, if and only if a measure μ is massive.

This Theorem was proved for continuous functions in [2] and for functions from the class $\mathcal{H}^p$, it is obtained by approximation of $f(z)$ by functions $f(r\zeta)$ when $r \to 1-0, r<1$, with respect to the metric of $\mathcal{H}^p$.

So, by Theorem 2.2 the series (2.1) converges absolutely for $\zeta \in \overline{D}$ and $z \in D$ and uniformly for $\zeta \in \overline{D}$ and $z \in K$, where K is an arbitrary compact subset in D.

Clearly $\partial D = \bigcup_{|\zeta|\in\partial D^+} \Delta_{|\zeta|}$. The following property of the Szegő kernel is evident:

$$h(\bar{\zeta}, z) = \overline{h(\zeta, \bar{z})} = h(z, \bar{\zeta}).$$

2.The Poisson kernel on n-circular domains

Let us recall the Poisson kernel

$$P(\zeta, z) = \frac{h(\bar{\zeta}, z)h(\zeta, \bar{z})}{h(\bar{z}, z)} = \frac{|h(\bar{\zeta}, z)|^2}{h(\bar{z}, z)}.$$

Note that the kernel $P(\zeta, z)$ is defined for $(\zeta, z) \in D \times D$, because $h(\bar{z}, z) > 0$.

Proposition 3.1. *If $f \in \mathcal{H}^p(D)$ $(p \geq 1)$, the following formula is true*

$$f(z) = \lim_{r\to 1} \frac{1}{(2\pi i)^n} \int_{\partial D^+} d\,\mu \int_{\Delta_{|\zeta|}} f\,(\zeta) P(\zeta, rz) \frac{d\zeta}{\zeta}, \qquad z \in D.$$

The proof follows from the form of the Poisson kernel and Theorem 2.2.

Lemma 3.2. *Consider the Szegő kernel at $\zeta = z$*

$h(\bar{z}, z) = \sum_{\alpha\geq 0} a_\alpha\, |z|^{2\alpha} > 0$

in D. Then $h(\bar{z}, z) \to \infty$, when $z \to \partial D$.

Suppose that a domain D satisfied the following property (A):

$h(\bar{\zeta}, rz)$ is uniformly bounded in z outside any neighborhood of ζ for $\zeta, z \in \partial D$ and $\zeta \neq z$, $r \to 1$.

Theorem 3.3. *Let D be a domain with the property (A) and $f \in \mathcal{L}^p(\partial D)$. Then the Poisson integral*

$F(z) = P[f](z) = \lim_{r\to 1} \frac{1}{(2\pi i)^n} \int_{\partial D^+} d\,\mu \int_{\Delta_{|\zeta|}} f\,(\zeta) P(\zeta, rz) \frac{d\zeta}{\zeta}$

is a real-analytic function on D and its values on the boundary with respect to the metric $\mathcal{L}^p$ coincides with f on ∂D.

Proof. Real-analyticity of $F(z)$ follows from the real-analyticity of the Szegő and Poisson kernels. By condition (A) and Lemma 3.2 we obtain that $P(\zeta, rz)$ uniformly converges to zero outside any neighborhood's at the point ζ for $\zeta, z \in \partial D$, $\zeta \neq z$ and $r \to 1$. Besides $P(\zeta, z) > 0$ and $P[1](\zeta) = 1$. Hence the Poisson kernel $P(\zeta, z)$ approximate identity [28, Page 49], where $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n$, $d\bar{\zeta}[k] = d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{k-1} \wedge d\bar{\zeta}_{k+1} \wedge \ldots \wedge d\bar{\zeta}_n$. The proof is complete. □

3.The modified Poisson kernel

Consider the following differential form

$$\omega = c \sum_{k=1}^{n} (-1)^{k-1}\, \bar{\zeta}_k\, d\bar{\zeta}[k]\, d\zeta,$$

where $c = \frac{(n-1)!}{(2\pi i)^n}$.

Let us find the restriction of this form on boundary ∂D of the domain

$$D = \{z \in \mathbb{C}^n\colon\ \rho(|z_1|^2, \ldots, |z_n|^2) < 0\},$$

where $\rho(z)$ is a twice smooth function and $\operatorname{grad}\rho = \left(\frac{\partial \rho}{\partial z_1}, \ldots, \frac{\partial \rho}{\partial z_n}\right) \neq 0$ on ∂D.

Denote $|z_k|^2 = t_k$, $k = 1, \ldots, n$. Then

$$\operatorname{grad}\rho = \left(\frac{\partial \rho}{\partial t_1}\bar{z}_1, \ldots, \frac{\partial \rho}{\partial t_n}\bar{z}_n\right) \neq 0$$

The function ρ can be choose such that $|\operatorname{grad}\rho||_{\partial D} = 1$. Let $\nu = \omega|_{\partial D}$, and in this case it is not hard to check that (see for example [15, Lemma 3.5]),

$$\nu = c \sum_{k=1}^{n} \bar{\zeta}_k \frac{\partial \rho}{\partial \bar{\zeta}_k} d\sigma = c \sum_{k=1}^{n} t_k \frac{\partial \rho}{\partial t_k} d\sigma,$$

where $d\sigma$ is the Lebesgue measure on ∂D. In a case of n-circular domain we have $d\sigma = d\sigma_+ \cdot d\sigma'$, where $d\sigma'$ is a measure defined by the form

$$\frac{1}{(2\pi i)^n} \frac{d\zeta_1}{\zeta_1} \wedge \ldots \wedge \frac{d\zeta_n}{\zeta_n},$$

and $d\sigma_+$ is the Lebesque measure on ∂D^+. Hence

$$\nu = c\sum_{k=1}^{n} t_k \frac{\partial\rho}{\partial t_k}\, d\sigma_+ \cdot d\sigma'.$$

Set

$\mu = c\sum_{k=1}^{n} t_k \frac{\partial\rho}{\partial t_k}\, d\sigma_+.$ (4.1)

Lemma 4.1. *If D is a complete n-circular domain, then μ is a measure on ∂D^+.*

The proof can be find in [22, Page 292].

Corollary 4.2. *If D is a complete strongly pseudoconvex n-circular domain, then μ is a massive measure on ∂D^+.*

Consider *the modified Poisson kernel*

$$Q(\zeta, z, w) = \frac{h(\bar\zeta, z)h(\zeta, w)}{h(w, z)}.$$

Then for $w = \bar\zeta$ we get $Q(\zeta, z, \bar z) = P(\zeta, z)$ and $h(\bar z, z) > 0$. Therefore there exists a neighboard U of the diagonal $w = \bar z$ in $D_z \times D_w$ such that $h(w, z) \neq 0$.

Consider a function

$$\Phi(z, w) = c\int_{\partial D} f(\zeta)Q(\zeta, z, w)d\nu =$$

$$= c\int_{\partial D^+} d\mu \int_{\Delta_{|\zeta|}} f(\zeta)Q(\zeta, z, w)\frac{d\zeta}{\zeta}, \qquad (z, w) \in D \times D.$$

This function is holomorphic in the variables $(z, w) \in U$, and for $w = \bar z$ function $\Phi(z, w) = F(z)$ and

$$\left.\frac{¶^{d+g}F(z,w)}{¶z^{d}¶w^{g}}\right|_{w=\bar z} = \frac{¶^{d+g}F(z)}{¶z^{d}¶\bar z^{g}} \qquad (4.2)$$

where

$$\frac{\partial^{\delta+\gamma}\Phi(z,w)}{\partial z^\delta\,\partial w^\gamma} = \frac{\partial^{\delta_1+\cdots+\delta_n+\gamma_1+\cdots\gamma_n}\Phi(z,w)}{\partial z_1^{\delta_1}\cdots\partial z_n^{\delta_n}\,\partial w_1^{\gamma_1}\cdots\partial w_n^{\gamma_n}},$$

$$\frac{\partial^{\delta+\gamma}F(z)}{\partial z^\delta\,\partial \bar z^\gamma} = \frac{\partial^{\delta_1+\cdots+\delta_n+\gamma_1+\cdots\gamma_n}F(z)}{\partial z_1^{\delta_1}\cdots\partial z_n^{\delta_n}\,\partial \bar z_1^{\gamma_1}\cdots\partial \bar z_n^{\gamma_n}},$$

and $\delta = (\delta_1, \dots, \delta_n)$, $\gamma = (\gamma_1, \dots, \gamma_n)$.

Let $\zeta = bt$, $b \in \mathbb{CP}^{n-1}$. As it was proved in [16] (see below also [15, Section 15])

$\omega = c\frac{dt}{t} \wedge \lambda(b),$ (4.3)

where $\lambda(b)$ is the differential form of type $(n-1, n-1)$ independent of t.

From now on we shall assume the existence of the direction $b^0 \neq 0$ such that

$\langle b^0, \bar\zeta\rangle \neq 0 \quad \zeta \in \overline{D}.$ (4.4)

Denote by L_G the set of all complex lines of the form

$\ell_{z,b} = \{\zeta \in \mathbb{C}^n\colon \zeta_j = z_j + b_j t,\ j = 1, \dots, n,\ t \in \mathbb{C}\},$ (4.5)

passing through a point $z\in\varGamma$ in the direction of vector $b \in \mathbb{CP}^{n-1}$ (the direction b is defined up to multiplication to a complex number $\lambda \neq 0$).

By Sard's Theorem for almost all points $z \in \mathbb{C}^n$ and for a fixed point $b \in \mathbb{CP}^{n-1}$ the intersection $\ell_{z,b} \cap \partial D$ consists a finite number of piecewise-smooth curves (beyond degenerate case when $\partial D \cap \ell_{z,b} = \emptyset$).

It is known that if $f \in \mathcal{L}^p(\partial D)$, $p \geq 1$, then for almost all $z \in D$ and almost all $b \in \mathbb{CP}^{n-1}$ the function $f \in \mathcal{L}^p(\partial D \cap \ell_{z,b})$ (see [24]).

We will say that a function $f \in \mathcal{L}^p(\partial D)$ has the *one-dimensional holomorphic extension property along the family* $l_{z,b}$ O L_G of the form (4.5), if for almost all lines $\ell_{z,b}$ such that $\partial D \cap \ell_{z,b} \neq \emptyset$ there exists a function f_ℓ with properties

1. $f_\ell \in \mathcal{H}^p(D \cap \ell_{z,b})$,
2. normal boundary values by the metric $\mathcal{H}^p$ of function f_ℓ coincides with f on $\partial D \cap \ell_{z,b}$ almost everywhere.

Consider the Bochner–Martinelli kernel

$$U(\zeta,z)=\frac{(n-1)!}{(2\pi i)^n}\sum_{k=1}^{n}(-1)^{k-1}\frac{\bar\zeta_k-\bar z_k}{|\zeta-z|^{2n}}\,d\bar\zeta[k]\wedge d\zeta,$$

where $d\zeta=d\zeta_1\wedge\ldots\wedge d\zeta_n$, and $d\bar\zeta[k]$ gets from $d\bar\zeta$ throwing the differential $d\bar\zeta_k$.

For a function $f\in\mathcal{L}^p(\partial D)$ define the Bochner–Martinelli integral by the following way

$F(z)=\int_{\partial D_\zeta}f(\zeta)U(\zeta,z),\qquad z\notin\partial D.$ (4.6)

A function $F(z)$ is harmonic outside of the boundary of domain and tends to zero when $|z|\to\infty$.

A subset $\mathfrak{L}_\Gamma$ is said to be *sufficient for holomorphic extension*, if the function $f\in\mathcal{L}^p(\partial D)$ has the one-dimensional holomorphic extension property along almost all complex lines from a family $\mathfrak{L}_\Gamma$, and then the function f extends holomorphically to D and belong the class $\mathcal{H}^p$.

From now on without loss of generality we assume that $0\in D$.

Theorem 4.3. *Let D be a bounded strongly convex n-circular domain and a function $f\in\mathcal{L}^p(\partial D)$ has the one-dimensional holomorphic extension property along complex lines passing through the origin. Then $\Phi(0,w)=$ const and* $\left.\frac{\P^d F(z,w)}{\P z^d}\right|_{z=0}$ *is a polynomial in w of degree not higher than $\|\delta\|$.*

Proof. Let $\ell_{0,b}$ be the line passing through the origin in the direction of a vector $b\in\mathbb{CP}^{n-1}$. Consider

$$Q(bt,z,w)=\frac{h(\bar b\bar t,z)h(bt,w)}{h(z,w)}.$$

Then

$h(\bar b\bar t,z)=\sum_{\alpha\ge0}a_\alpha\left(\bar bz\right)^\alpha\bar t^{\|\alpha\|}$ (4.7)

and $h(0,0)=h(\zeta,0)=a_0$. Thus

$$\Phi(0,0)=\int_{\partial D}f(\zeta)\frac{h(\bar\zeta,0)h(\zeta,0)}{h(0,0)}\,dv=$$
$$=\frac{1}{h(0,0)}\int_{\partial D}f(\zeta)h(\bar\zeta,0)h(\zeta,0)\,dv=$$
$$=\frac{c}{h(0,0)}\int_{\partial D\cap l_{0,b}}\lambda(b)\int_{l_{0,b}}h\left(\bar b\bar t,0\right)h(bt,0)\frac{f(bt)}{t}\,dt=$$
$$=\frac{ca_0^2}{a_0}\int_{\partial D\cap l_{0,b}}\lambda(b)\int_{l_{0,b}}\frac{f(bt)}{t}\,dt=ca_0\int_{\partial D\cap l_{0,b}}\lambda(b)\int_{l_{0,b}}\frac{f(bt)}{t}\,dt.$$

Let us consider derivatives

$$\frac{\partial^{\delta+\gamma}\Phi(z,w)}{\partial z^\delta\,\partial w^\gamma}=\frac{\partial^{\delta_1+\cdots+\delta_n+\gamma_1+\cdots\gamma_n}\Phi(z,w)}{\partial z_1^{\delta_1}\cdots\partial z_n^{\delta_n}\,\partial w_1^{\gamma_1}\cdots\partial w_n^{\gamma_n}},$$

where $\delta=(\delta_1,\ldots,\delta_n)$, $\gamma=(\gamma_1,\ldots,\gamma_n)$. We have

$$\frac{\partial^\gamma\Phi(0,w)}{\partial w^\gamma}\int_{\partial=D}f(\zeta)\frac{\partial^\gamma Q(\zeta,0,w)}{\partial w^\gamma}\,dv=$$
$$=\ \frac{a_0}{a_0}\int_{\partial D}f(\zeta)\frac{\partial^\gamma h(\zeta,w)}{\partial w^\gamma}\,dv=\int_{\partial D}f(\zeta)\frac{\partial^\gamma h(\zeta,w)}{\partial w^\gamma}\,dv.$$

Now compute

$$\frac{\partial^\gamma h(\zeta,w)}{\partial w^\gamma}=\sum_{\alpha-\gamma\ge0}a_\alpha\,d_{\alpha,\gamma}\zeta^\alpha w^{\alpha-\gamma},$$

where $d_{\alpha,\gamma}$ are constants. Then for $w=0$ we get

$$\left.\frac{\P^g h(z,w)}{\P w^g}\right|_{w=0}=a_a d_{g,g}z^g.$$

Thus we obtain that

$$\left.\frac{\partial^{\gamma}\Phi(0,w)}{\partial w^{\gamma}}\right|_{w=0} = c\int_{\partial D\cap \ell_{0,b}} \lambda(b)\int_{\ell_{0,b}} f(bt)a_{\gamma}d_{\gamma,\gamma}b^{\gamma}t^{\|\gamma\|}\frac{dt}{t} = 0$$

for $\| \gamma \| > 0$. This means that $\Phi(0,w) = \text{const}$.

Compute

$$\left.\frac{\partial^{\delta+\gamma}(h(\bar{z},z)\cdot h(z,w))}{\partial z^{\delta}\partial w^{\gamma}}\right|_{\substack{z=0\\ w=0}} = \frac{\partial^{\delta}h(\bar{z},z)}{\partial z^{\delta}}\cdot\left.\frac{\partial^{\gamma}h(z,w)}{\partial w^{\gamma}}\right|_{\substack{z=0\\ w=0}} = \delta!a_{\delta}\bar{z}^{\delta}\gamma!a_{\gamma}z^{\gamma}$$

where $\delta! = \delta_1!\cdot\ldots\cdot\delta_n!$, $\gamma! = \gamma_1!\cdot\ldots\cdot\gamma_n!$. Then

$$\left.\frac{\partial^{\delta+\gamma}Q(z,z,w)}{\partial z^{\delta}\partial w^{\gamma}}\right|_{\substack{z=0\\ w=0}} = \left.\frac{\partial^{\delta+\gamma}(h(\bar{z},z)\cdot h(z,w))}{\partial z^{\delta}\partial w^{\gamma}}\right|_{\substack{z=0\\ w=0}} = \sum_{k=1}^{N} C_k\bar{z}^{\delta-e^k}z^{\gamma-e^k},$$

because by substituting $z = 0$ and $w = 0$ in derivatives, we can see that derivatives in z and w with different order are equal to zero.

Let $\| \delta \| < \| \gamma \|$. Then from (4.3) and (4.8) we get that

$$\left.\frac{\partial^{\delta+\gamma}\Phi(z,w)}{\partial z^{\delta}\partial w^{\gamma}}\right|_{\substack{z=0\\ w=0}} = c\int_{\partial D}\left.\frac{\partial^{\delta+\gamma}Q(z,z,w)}{\partial z^{\delta}\partial w^{\gamma}}\right|_{\substack{z=0\\ w=0}} d\nu =$$

$$= c\sum_{k=1}^{N} C_k\int_{\partial D} f(z)\bar{z}^{\delta-e^k}z^{\gamma-e^k}d\nu = c\sum_{k=1}^{N} C_k\int_{\partial D\cap\ell_{0,b}}\lambda(b)\int_{\ell_{0,b}} f(bt)\bar{t}^{\|\delta-e^k\|}t^{\|\gamma-e^k\|}dt =$$

$$= c\sum_{k=1}^{N} C_k\int_{\partial D\cap\ell_{0,b}}\lambda(b)\int_{\ell_{0,b}} f(bt)\bar{t}^{\|\delta-e^k\|}t^{\|\gamma-e^k\|-1}dt =$$

$$= C_{\delta}\sum_{k=1}^{N} C_k\int_{\partial D\cap\ell_{0,b}}\lambda(b)\int_{\ell_{0,b}} f(bt)t^{\|\gamma-e^k\|-1}dt = 0$$

because the intersection of D and $\ell_{0,b}$ is a disc. So, derivatives $\left.\frac{\partial^{\delta}\Phi(z,w)}{\partial z^{\delta}}\right|_{z=0}$ are polynomials in w of degree not higher than $\| \delta \|$. The proof is complete. □

For continuous functions Theorem 4.3 was proved in [21].

4.Main results

Let us first construct a map $\zeta = \chi(\eta)$: $\overline{B} \to \overline{D}$, where B is the unit ball in $\mathbb{C}^n$ with the center at the origin, mapping the origin to a point $a \in D$. The map χ will be construct by the following way:

Consider complex lines $\lambda_b = \{\eta \in \mathbb{C}^n\colon \eta = b\tau,\ \tau \in \mathbb{C}\}$ and $\ell_{a,b} = \{\zeta \in \mathbb{C}^n\colon \zeta = a + bt,\ t \in \mathbb{C}\}$, where $b \in \mathbb{CP}^{n-1}$. The intersection $D_{a,b} = \ell_{a,b} \cap D$ is a strongly convex domain in $\mathbb{C}$, and therefore there exists a conformal map $\chi_b(\tau)$ of unit ball in $\mathbb{C}$ into $D_{a,b}$, sending the point $\tau = 0$ to the point $t = 0$. By Caratheodory's Theorem [27, Page 228] this map can be extended to the homeomorphism of closed domains. Then to the point $\eta = b\tau \in D \cap \lambda_b$ we put the point $\chi(\eta) = a + b\chi_b(\tau) \in D_{a,b}$. We will use Lemmata 3 and 4 from [21].

Lemma 5.1. *Let D be a bounded strongly convex n-circular domain. Then the map $\chi(\eta)$ is a well defined diffeomorphism from $\overline{B}$ onto $\overline{D}$ in the class $\mathcal{C}^1$.*

From now on we assume that D is a bounded strongly convex n-circular domain with twice smooth boundary.

Lemma 5.2. *The derivatives of $\chi(\eta)$ is holomorphic in τ for fixed b.*

Lemma 5.3. *Let $f \in \mathcal{L}^p(\partial D)$ be a function with the one-dimensional holomorphic extension property along almost all complex lines passing through the point $a \in D$. Then the function $f^*(\eta) = f(\chi(\eta)) \in \mathcal{L}^p(\partial B)$ and it has the one-dimensional holomorphic extension property along almost all complex lines passing from the origin.*

Proof. Consider a holomorphic extension $f_{a,b}(\zeta)$ of the function f on $D_{a,b}$. Then the function $f_b^*(\eta) = f_{a,b}(\chi_b(\tau))$ is holomorphic in τ in $B \cap \lambda_b$ by the construction of $\chi(\eta)$. The proof is complete. □

Making the change in the integral for a function Φ, we get

$$\Phi(z,w) = \int_{\partial D} f(\zeta) Q(\zeta,z,w)\, d\nu(\zeta) =$$

$$= \int_{\partial B} f(\chi(\eta)) Q(\chi(\eta),z,w)\, d\nu(\chi(\eta)) = \int_{\partial B} f^*(\eta) Q^*(\eta,z,w)\, d\nu^*(\eta).$$

Consider the following form

$$\omega^*(\eta) = \omega(\chi(\eta)) = \sum_{k=1}^{n} (-1)^{k-1}\, \bar{\chi}_k(\eta)\, d\bar{\chi}(\eta)[k] \wedge d\chi(\eta).$$

By Lemma 5.2 the form $d\chi(b\tau)$ is a holomorphic function in τ for fixed b, and the form $d\bar{\chi}(b\tau)[k]$ is antiholomorphic in τ for fixed b.

Lemma 5.4. *Forms $d\bar{\chi}(b\tau)|_{|\tau|=1}$, $k = 1, \dots, n$ are forms with holomorphic coefficients in τ.*

Proof. Since a function $\chi_k(b\tau)$ is conformal in τ for fixed b, it follows that its derivative in τ does not equal zero. Then a function $\overline{\chi_k}(b\tau)$ is antiholomorphic in τ for fixed b and

$$\overline{\chi_k}(b\tau) = \bar{\tau}\overline{\psi_k}(b\tau),\ \overline{\psi_k}(b\tau)|_{\tau=0} \neq 0.$$

For $|\tau| = 1$ we have $\bar{\tau} = \frac{1}{\tau}$ and therefore

$$\overline{\chi_k}(b\tau)|_{|\tau|=1} = \frac{1}{\tau}\overline{\psi_k}(b\tau) = \frac{1}{\tau}\psi_k(\bar{b}\bar{\tau}) = \frac{1}{\tau}\psi_k\left(\bar{b}\frac{1}{\tau}\right).$$

Therefore the right side has a pole of the first order in $\tau = 0$. Then the form $d\overline{\chi_k}(b\tau)|_{|\tau|=1}$ coincides with a form with holomorphic coefficients at τ. The proof is complete. □

Lemma 5.5. *Let $f \in \mathcal{L}^p(\partial D)$ be a function with the one-dimensional holomorphic extension property along complex lines passing through point $a \in D$. Then*

$$\left.\frac{\partial^{\gamma}\Phi(z,w)}{\partial w^{\gamma}}\right|_{\substack{z=a\\ w=\bar{a}}} = 0$$

for $\|\gamma\| > 0$.

Proof. Consider a derivative

$$\left.\frac{\partial^{\gamma}\Phi(z,w)}{\partial w^{\gamma}}\right|_{w=\bar{a}} = \left.\int_{\partial B} f^*(\eta)\frac{\partial^{\gamma}Q^*(\eta,z,w)}{\partial w^{\gamma}}\sum_{k=1}^{n}(-1)^{k-1}\bar{\chi}_k(\eta)d\bar{\chi}(\eta)[k]\wedge d\chi(\eta)\right|_{w=\bar{a}} =$$

$$= \left.\int_{\partial B\cap\lambda_b}\int_{\lambda_b} f^*(b\tau)\frac{\partial^{\gamma}Q^*(b\tau,z,w)}{\partial w^{\gamma}}\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{\tau}\psi_k\left(\bar{b}\frac{1}{\tau}\right)d\chi\left(\bar{b}\frac{1}{\tau}\right)[k]\wedge d\chi(b\tau)\right|_{w=\bar{a}}.$$

We have

$$\left.\frac{\partial^{\beta}h(\chi(\eta),w)}{\partial w^{\beta}}\right|_{w=\bar{a}} = \sum_{\substack{\alpha-\beta\geq 0\\ \|\alpha-\beta\|\neq 0}} a_{\alpha}d_{\beta}\chi^{\alpha}\bar{a}^{\alpha-\beta},$$

where d_β are constants. Then

$$\left.\frac{\partial^\gamma Q^*(h,z,w)}{\partial w^\gamma}\right|_{\substack{z=a\\ w=\bar a}} = h(\hbar,a)\left.\frac{\partial^\gamma}{\partial w^\gamma}\left(\frac{h(z,w)}{h(z,w)}\right)\right|_{\substack{z=a\\ w=\bar a}} =$$

$$= h(\hbar,a)\sum_{0\le\beta\le\gamma} c_\beta \left.\frac{\frac{\partial^\beta h(z,w)}{\partial w^\beta}\cdot\frac{\partial^{\gamma-\beta}h(z,w)}{\partial w^{\gamma-\beta}}}{h^{\|\gamma\|+1}(z,w)}\right|_{\substack{z=a\\ w=\bar a}} =$$

$$= \frac{h(\hbar,a)}{h^{\|\gamma\|+1}(a,\bar a)}\sum_{0\le\beta\le\gamma} c_\beta \left.\frac{\partial^\beta h(z,w)}{\partial w^\beta}\cdot\frac{\partial^{\gamma-\beta}h(w,a)}{\partial w^{\gamma-\beta}}\right|_{w=\bar a}.$$

Therefore the right side of $\left.\frac{\partial^\gamma \Phi(z,w)}{\partial w^\gamma}\right|_{w=\bar a}$ is a linear combination of forms

$$\int_{\partial B\cap l_\beta}\int_{l_\beta} f^*(bt)h(\frac{\bar b}{t},a)\frac{\partial^\beta h(bt,w)}{\partial w^\beta}\cdot\left.\frac{\partial^{\gamma-\beta}h(w,a)}{\partial w^{\gamma-\beta}}\right|_{w=\bar a}\wedge$$

$$\wedge\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{t}\psi_k(\frac{\bar b}{t})d\chi(\frac{\bar b}{t})[k]\wedge d\chi(bt).$$

Since by Lemma 5.4 the form $d\bar\chi(b\tau)|_{|\tau|=1}$ has holomorphic coefficients in τ and f is a function with the one-dimensional holomorphic extension property along complex lines passing through point $a\in D$ we have

$$\int_{\partial B\cap l_\beta}\int_{l_\beta} f^*(bt)\frac{1}{t}h(\frac{\bar b}{t},a)\sum_{\substack{\alpha-\beta\ge 0\\ \|\alpha-\beta\|\ne 0}} a_\alpha d_\beta b^\alpha t^\alpha \bar a^{\alpha-\beta}\left.\frac{\partial^{\gamma-\beta}h(w,a)}{\partial w^{\gamma-\beta}}\right|_{w=\bar a}\wedge$$

$$\wedge\sum_{k=1}^{n}(-1)^{k-1}\psi_k(\frac{\bar b}{t})d\chi(\frac{\bar b}{t})[k]\wedge d\chi(bt) = 0.$$

for $\|\gamma\|>0$. The proof is complete. □

Corollary 5.6. *Under the hypotheses of Lemma 5.5 the function $\Phi(a,w)=\text{const}$.*

Theorem 5.7. *Let D be a complete strongly convex n-circular domain with the twice smooth boundary and let $f(\zeta)\in\mathcal{L}^p(\partial D)$, $a,c\in D$. Suppose that the function $\Phi(z,w)$ satisfied the conditions: $\Phi(a,w)=\text{const}$, $\Phi(c,w)=\text{const}$, and $\frac{\partial^\alpha\Phi(a,w)}{\partial z^\alpha}$, $\frac{\partial^\alpha\Phi(c,w)}{\partial z^\alpha}$ are polynomials in w of degree not higher than $\|\alpha\|$. Then for any fixed point z on the complex plane $\ell_{a,c}=\{(z,w): z=at+c(1-t),\ w=\bar a t+\bar c(1-t),\ t\in\mathbb{C}\}$ the equality $\Phi(z,w)=\text{const}$ in w, i.e., $\frac{\partial^\gamma\Phi(z,w)}{\partial w^\gamma}=0$ at $\|\gamma\|>0$.*

The proof of this Theorem repeated the proof of Theorem 3 from [22].

Corollary 5.8. *Under the hypotheses of Theorem 5.7 the equality* $\left.\frac{\partial^\gamma F(z)}{\partial z^\gamma}\right|_{z=at+(1-t)c} = 0$ *holds if $\|\gamma\|>0$.*

Theorem 5.9. *Let $n=2$ and let $f\in\mathcal{L}^p(\partial D)$ be a function with one-dimensional holomorphic extension property along a family of lines $\mathfrak{L}_{a,c,d}$ and let points $a,c,d\in D$ do not lie on complex line in $\mathbb{C}^2$. Then $\frac{\partial^\gamma\Phi(z,w)}{\partial w^\gamma}=0$ for every $z\in D$ and $\|\gamma\|>0$, and hence a function f extends holomorphic in D and an extension lies in $\mathcal{H}^p$.*

Proof. Let $\tilde z$ be an arbitrary point from $\ell_{a,c}$. By Theorem 5.7 we obtain that

$$\frac{\partial^\gamma\Phi(\tilde z,w)}{\partial w^\gamma}=0 \qquad (5.1)$$

for $\| \gamma \| > 0$. We connect point $\tilde{z}$ with the point d by a line $\ell_{\tilde{z},d}$ and again applying Theorem 5.7 for a point $\tilde{\tilde{z}} \in \ell_{\tilde{z},d}$, we get that $\frac{\partial^\gamma \Phi(\tilde{\tilde{z}},w)}{\partial w^\gamma} = 0$ for $\| \gamma \| > 0$. Therefore for any point $\tilde{z}$ from some open subset the condition (5.1) holds.

Now putting in equality (5.1) $w = \bar{z}$ and using equality (4.2), we obtain that $\frac{\partial^\gamma F(z)}{\partial \bar{z}^\gamma} = 0$ in some open subset in D. According to analyticity of the function $F(z)$ we imply that $\frac{\partial F(z)}{\partial \bar{z}_j} = 0$ for all $z \in D$ and $j \in \{1, \dots, n\}$. Since by Theorem 3.3 the equality $F(\zeta)|_{\partial D} = f(\zeta)$, the function $f(\zeta)$ is holomorphic in D. The proof is complete. □

Denote by $\mathfrak{A}$ the set of points $a_k \in D \subset \mathbb{C}^n$, $k = 1, \dots, n+1$, do not lie on complex hyperplane in $\mathbb{C}^n$.

The following is the main result of this paper.

Theorem 5.10. *Let $f \in \mathcal{L}^p(\partial D)$ has the one-dimensional holomorphic extension property along a family of lines $\mathfrak{L}_{\mathfrak{A}}$. Then $\frac{\partial^\gamma \Phi(z,w)}{\partial w^\gamma} = 0$ for all $z \in D$ and $\| \gamma \| > 0$, Moreover, a function f extends holomorphically in D and an extension lie in the class $\mathcal{H}^p$.*

Proof. The proof is by induction on n. The basis of induction is Theorem 5.9 $(n = 2)$. Suppose that for any $k < n$ Theorem is true. Consider a complex plane Γ, passing through points $a_1, \dots, a_n$, its dimension is $n - 1$ and $a_{n+1} \notin \Gamma$. The intersection $\Gamma \cap D$ is a strongly convex domain in $\mathbb{C}^{n-1}$. Now the restriction $f|_{\Gamma \cap \partial D}$ of f is integrable and has the one-dimensional holomorphic extension property along a family of lines $\mathfrak{L}_{\mathfrak{A}_1}$, where $\mathfrak{A}_1 = \{a_1, \dots, a_n\}$. By the assumption of induction we have $\frac{\partial^\gamma \Phi(z\prime,w)}{\partial w^\gamma} = 0$ for $\| \gamma \| > 0$ and for all $z' \in \Gamma \cap D$.

Now connecting points $z' \in \Gamma$ with the point a_{n+1}, by Theorem 5.9, we get that $\frac{\partial^\gamma \Phi(z,w)}{\partial w^\gamma} = 0$ for some open subset in $D \times D$ for all $\| \gamma \| > 0$. Thus, analogously to Theorem 5.9, we have that $F(z)$ is holomorphic in D, and therefore a function f extends holomorphically in D. The proof is complete. □

We discuss a general result of holomorphic extension of a real analytic function f defined on the boundary ∂D of a real analytic strictly convex subset $D \subset\subset \mathbb{C}^n$. We show that this follows from the hypothesis of separate holomorphic extension along stationary/extremal discs.

We show that moments on vertical and horizontal affine slices of certain perturbations of the boundary of the ball are sufficient to detect the existence of a holomorphic extension to the interior. For one domain, the result is proved for L^∞ functions.

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