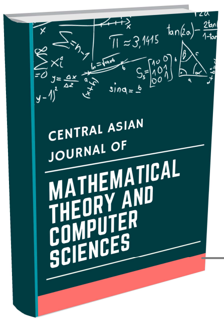


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# Nonsmooth Optimal Control Problem for System with Delay under Conditions of Indeterminacy


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**Abstract:**

The paper considers a mathematical model of control system with delay under conditions of uncertainty external influence. Optimization problem for nonsmooth terminal functional in the minimum type is researched. The necessary and sufficient conditions of optimality are obtained. Construction algorithm of optimal control is suggested.



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## 1. Introduction.

Nonsmooth optimal control problems constitute an important direction in optimization theory. They arise when modeling many problems from economics, technology and production. Such optimal control problems are studied using the methods of convex and nonsmooth analysis [1–3]. Methods of multivalued analysis and the mathematical apparatus of differential inclusions play an important role in the development of this direction [4–6].

Functions defined through the operation of minimization or maximization with respect to some parameters belong to a wide class of nonsmooth functions [3]. Optimization of such functions in control theory occurs when the model takes into account the inaccuracies and errors of various external parameters and initial data. In such models, when information about undefined parameters and data is minimal, i.e. only the sets of their possible values are known, the principle of obtaining a guaranteed result is applied [7,8]. This principle leads to the problems of minimizing functionals defined through the operations of maximum or minimum.

Questions of constructing a control action taking into account such important factors as inaccuracies in the initial data, incomplete information on the parameters of external influences, etc. lead to models of control and observation under conditions of uncertainty (incompleteness of information). In studies of such models, problems of controlling ensembles of trajectories, methods of forecasting and estimating the phase state, conditions of guaranteed control, and problems of minimax synthesis are of great interest[8, 9,10]. It should be noted that these problems are actual for models of control systems with delays. Models of control systems with delays have applications in the theory of automatic control, in the study of problems of long-term forecasting in economics and problems from other fields of science and technology.

## 2. Statement of the problem. Research methods.

Consider a mathematical model of a dynamic control system with delays of the form

$$\frac{dx(t)}{dt} = A(t)x(t) + \sum_{i=1}^{k} A_i(t)x(t-h_i) + B(t)u(t) + q(t), \quad t \in T = [t_0, t_1], \quad (1)$$

$$x(t) = \varphi_0(t),\ t \in T_0 = [t_0 - h_0, t_0],\ h_0 = \max_{i=\overline{1,k}} h_i, \quad (2)$$

where $x \in R^n$, $A(t)$, $A_i(t)$ - $n \times n$ -matrices, $B(t)$ - $n \times m$ -matrix, $u(t)$ - $m$-control vector, $q(t)$ - $n$-vector of external influence, $\varphi_0(\cdot) \in C^n(T_0)$. Suppose that in the system consideration the parameter $q(t)$ is not controlled by the controlling party and its values are not known in advance.

Suppose that the function $q(t)$, $t \in T$, is summable and a geometric constraint of the form $q(t) \in W(t)$, $t \in T$, is satisfied, where $W(t)$ is a convex compact from $R^n$. We denote the set of all such functions $q(\cdot)$ by $Q(T)$.

In control system (1), the set of admissible controls *U(T)* consists of all measurable bounded *m*-vector functions $u = u(t)$ satisfying the condition $u(t) \in V$ for almost all $t \in T = [t_0, t_1]$, where $V \subset R^m$ is a convex compact set.

We will assume that the following conditions are met:

a) elements of matrices $A(t)$, $A_i(t), i = \overline{1,k}$, and $B(t)$ are summable on the segment $T = [t_0, t_1]$;

b) the multivalued mapping $t \to W(t)$ is measurable on *T*, and $\|\xi\| \le q_0(t)$, $\forall \xi \in W(t)$, where $q_0(\cdot) \in L_1(T)$.

Then, by virtue of the results of the theory of differential equations, for each admissible control $u = u(t)$, $t \in T$, and function $q(\cdot) \in L_1(T)$ there exist an absolutely continuous solution $x(t)$, $t \in T$, of system (1) – (2). Such a solution $x(t) = x(t, u, q)$ will be called the controlled motion (trajectory) of this system.

Let the quality of each controlled motion $x(t) = x(t, u, q)$ be estimated by the terminal functional

$$J(x(\cdot, u, q)) = g(x(t_1, u, q)),\ g(x) = \min_{z \in Z}(Px, z),$$

where $P$ is a $l \times n$ -matrix, $Z \subset R^l$ is a compact.

Taking into account the uncertainty of the parameter of external influences $q(t)$ in the control system (1), we will optimize the terminal functional according to the minimax principle, i.e. consider the problem

$$\sup_{q \in Q(T)} g(x(t_1, u, q)) \to \min_{u \in U(T)}. \quad (3)$$

So, it is required to find a control $u^* \in U(T)$ satisfying the condition

$$\min_{u \in U(T)} \sup_{q \in Q(T)} \min_{z \in Z}(Px(t_1, u, q), z) = \sup_{q \in Q(T)} \min_{z \in Z}(Px(t_1, u^*, q), z). \quad (4)$$

Such a control $u^*(t), t \in T$ is called the optimal control in the minimax control problem (3).

It easily follows from the results of the theory of multivalued mappings and differential inclusions that the set of controlled motions $x(t) = x(t, u, q)$ of system (1) - (2) coincides with the set of absolutely continuous solutions of the Cauchy problem for a differential inclusion:

$$\frac{dx}{dt} \in A(t)x(t) + \sum_{i=1}^{k} A_i(t)x(t-h_i) + B(t)u(t) + W(t),\ t \in T, \quad (5)$$

$$x(t)=\varphi_0(t),\ t\in T_0=[t_0-h_0,t_0]. \qquad (6)$$

Thus, controlled differential inclusion (5) describes the dynamics of control system (1) when the parameter of external influence is inaccurate, and $q(t)\in W(t)$, $t\in T$.

Let: $H(u)$ be the set of absolutely continuous solutions $x=x(t)$, $t\in T$ of the Cauchy problem (5) – (6); $X(t,u)=\{\xi\in R^n:\xi=x(t),x(\cdot)\in H(u)\}$, $t\in T$, - the corresponding ensemble of trajectories. So, we have:

$$H(u)=\{x(\cdot,u,q):q\in Q(T)\},\ X(t,u)=\{\xi\in R^n:\xi=x(t_1,u,q),q\in Q(T)\}.$$

Thus, the minimax terminal control problem (3) is reduced to the minimax problem of controlling the terminal state of an ensemble of trajectories of a differential inclusion (5) with the initial condition (6):

$$\sup_{\xi\in X(t_1,u)}\min_{z\in Z}(Px,z)\to\min_{u\in U(T)}. \qquad (7)$$

Optimal control $u^0(t)$, $t\in T$, in problem (7) is optimal in minimax problem (3).

According to the results of work [11], the set $H(u)$ is non-empty, convex and compact in $C^n(T_1)$, where $T_1=[t_0-h_0,t_1]$, and the set $X(t,u)$ is a convex compact in $R^n$ for all $t\in T$. By virtue of the results of [12], we have:

$$X(t_1,u)=S(t_1)+\int_{t_0}^{t_1}F(t_1,\tau)B(\tau)u(\tau)d\tau+\int_{t_0}^{t_1}F(t_1,\tau)W(\tau)d\tau, \quad (8)$$

where $F(t,\tau)$ - $n\times n$ is a matrix function satisfying the equation

$$\frac{\partial F(t,\tau)}{\partial\tau}=-F(t,\tau)A(\tau)-F(t,\tau+h)A_1(\tau+h),\ \tau\le t,$$

$$F(t,t-0)=E,\ F(t,\tau)\equiv 0,\ \tau\ge t+0, \quad S(t_1,\varphi_0)=F(t_1,t_0)\varphi_0(t_0)+\sum_{i=1}^{k}\int_{t_0}^{t_0+h_i}F(t_1,\tau)A_i(\tau)\varphi_0(\tau-h_i)d\tau$$

## 3. Main results.

It easily follows from representation (8) that the support function $C(X(t_1,u),\psi)$ of a compact set $X(t_1,u)$ is expressed by the equality

$$C(X(t_1,u),\psi)=(S(t_1,\varphi_0),\psi)+\int_{t_0}^{t_1}(F(t_1,t)B(t)u(t),\psi)dt+\int_{t_0}^{t_1}C(F(t_1,t)W(t),\psi)dt. \qquad (9)$$

Now, using (9) and the minimax theorem from convex analysis [1], we obtain that for the functional

$$\Phi(u)=\sup_{\xi\in X(t_1,u)}\min_{z\in Z}(z,P\xi),\ u\in U(T)$$

the formula takes place:

$$\Phi(u)=\min_{z\in coZ}[(S(t_1),P'z)+\int_{t_0}^{t_1}(F(t_1,t)B(t)u(t),P'z)dt+\int_{t_0}^{t_1}C(F(t_1,t)W(t),P'z)dt], \qquad (10)$$

where *coZ* is the convex hull of the set *Z*.

Let's introduce the functional

$$\rho(z,u) = (S(t_1,\varphi_0), P'z) + \int_{t_0}^{t_1} (F(t_1,t)B(t)u(t), P'z)dt + \int_{t_0}^{t_1} C(F(t_1,t)W(t), P'z)dt \ .$$

Then formula (10) can be written as

$$\Phi(u) = \min_{z\in coZ} \rho(z,u), \quad u \in U(T).$$

Consider also the function

$$\mu(z) = (S(t_1,\varphi_0), P'z) + \int_{t_0}^{t_1} \min_{v\in V}(F(t_1,t)B(t)v, P'z)dt + \int_{t_0}^{t_1} C(F(t_1,t)W(t), P'z)dt. \qquad (11)$$

Then, it is easy to see that $\min\limits_{u\in U(T)} \rho(z,u) = \mu(z)$ and, therefore,

$$\min_{u\in U(T)} \Phi(u) = \min_{z\in coZ} \min_{u\in U(T)} \rho(z,u) = \min_{z\in coZ} \mu(z). \qquad (12)$$

All these considerations show that the results of [11] can be applied. Let $u^* = u^*(t)$, $t \in T$, be the optimal control in problem (7), and $z^* \in coZ$ be point of the global minimum of the function $\rho(z,u^*)$. Then:

$$\rho(z^*,u^*) = \min_{u\in U(T)} \rho(z^*,u).$$

Therefore, for almost all $t \in T$ we have

$$(F(t_1,t)B(t)u^*(t), P'z^*) = \min_{v\in V}(F(t_1,t)B(t)v, P'z^*). \qquad (13)$$

Now, using (12), we obtain that $z^*$ is the point of the global minimum of the function $\mu(z)$ on *coZ.*

Let, inversely, $z^* \in coZ$ be the point of the global minimum of the function $\mu(z)$, a $u^*(t)$, $t \in T$, is determined from (13). Then, based on formula (10), it can be shown that $\Phi(u) \geq \Phi(u^*), \forall u \in U(T)$, i.e. $u^*(t)$, $t \in T$, - optimal control in problem (7). Since problem (3) is reduced to problem (7), then, we can say that the following was obtained

**Theorem.** For the optimality of the control $u^*(t)$, $t \in T$, in problem (3) it is necessary and sufficient the existence of a vector $z^* \in coZ$, which is a point of the global minimum of a function $\mu(z)$ of the form (11), and the fulfillment of the condition (13) for almost all $t \in T$.

Using the obtained optimality conditions, we can propose the following algorithm for constructing the optimal control in problem (3).

Step 1. Find $n \times n$ - matrix function $F(t_1,\tau)$ and vector $S(t_1)$, which are used in formula (8);

Step 2. Determine the function $\mu(z)$ according to the formula (11);

Step 3. Find $z^*$ - the point of the global minimum of the function $\mu(z)$ on the set $coZ$;

Step 4. Find a function $u^* = u^*(t)$, $t \in T$ satisfying condition (13) on *T.*

Found after these steps, control $u^* = u^*(t)$, $t \in T$, will be optimal in problem (3).

**Example.** Consider a control system with delay and parameter of external influences $q(t) \in W = [-1,1]$:

$\dot{x}_1 = -x_2(t-1) + q(t),\ \dot{x}_2 = u(t),\ \ |u(t)| \le 1,\ \ q(t) \in W = [-1,1],\ t \in [0,2].$

The initial conditions are

$x_1(t) = \varphi_{01}(t) = 1,\ x_2(t) = \varphi_{02}(t) = t,\ -1 \le t \le 0.$

Let the quality of the control process be assessed by the terminal functional

$J = g(x(2) = \min_{z \in Z}(z, Px(2)),$

where $P = (1,\ 1)$, $Z = [-1,1]$.

According to the above algorithm, for the system under consideration, we have:

$$F(2,\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ 1 \le \tau \le 2;\ F(2,\tau) = \begin{pmatrix} 1 & \tau - 1 \\ 0 & 1 \end{pmatrix},\ 0 \le \tau \le 1;$$

$$S(2) + \int_0^2 F(2,\tau) W d\tau = \left[-\frac{1}{2}, \frac{7}{2}\right] \times \{0\};$$

$$\mu(z) = (S(2), P'z) + \int_0^2 \min_{|v| \le 1}(F(2,\tau)Bv, P'z)dt + \int_0^2 C(F(2,\tau)W, P'z)d\tau = \frac{1}{2}|z| + \frac{3}{2}z.$$

$z^* = -1$ there is a point of the global minimum of the function $\mu(z)$ on $Z = [-1,1]$. Then, by virtue of the minimum condition $\min_{|v| \le 1}(F(2,t)Bv, P'z^*) = (F(2,\tau)Bu^*, P'z^*)$, we obtain the optimal control $u^*(t) \equiv 1$.

## 4. Conclusion.

The paper considers the problem of terminal control for a model of a control system with a delay under conditions of inaccuracy of information about the parameter of external influences. In the studied problem, the terminal functional is distinguished by its nonsmooth character. The terminal functional is specified through the operation of a minimum with respect to a parameter with values from a compact set.

To study this optimal control problem, the mathematical apparatus of differential inclusions and nonsmooth analysis are used. As a result, the problem of minimax control of the terminal state of a differential inclusion is obtained. Using control methods for differential inclusions, necessary and sufficient conditions for optimality are indicated. These conditions make it possible to reduce the infinite-dimensional optimal control problem to the finite-dimensional optimization problems.

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